Lecture Note for Fourier analysis Seminar

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1 5.3 The Poisson summation formula

The definition of the Fourier transform was motivated by the desire for a continuous version of Fourier series, applicable to functions defined on the real line.

Aim : Show that there exists a further remarkable connection between the analysis of functions on the circle and related functions on $\mathbb{R}$. (The relationship between Fourier series and Fourier analysis) Wish to construct a "period version" form of f)?

1.1 *Prerequisite

$\mathcal{S}(\mathbb{R})$: (the **Schwartz space** on $\mathbb{R}$ consists of the set of all indefinitely differentiable functions f and all its derivatives $f^{'}, f^{''}, \cdots, f^{(l)}, \cdots$ are rapidly decreasing $\sup_{x\in\mathbb{R}}|x|^k|f^{(l)}(x)| < \infty$, $\forall k, l \geq 0$)

***function on a circle**:$f(\theta) = F(e^{i\theta})$, which means it is periodic on $\mathbb{R}$ of period 2π, it almost preserves continuous and differentiable. It is equivalent between 2π-periodic functions and function on the circle.

1.2 Poisson summation formula

The construct of periodization
Given a function $f \in \mathcal{S}(\mathbb{R})$ on a **real line**, we can construct a new function **on the circle** by the recipe

$$F_1(x) = \sum_{n=-\infty}^{\infty} f(x+n),$$

F_1 is called the **periodization** (or **periodic summation**) of f.

$\underline{Property1:}$ F_1 is continuous.
pf. The function is indefinitely differentiable, so they are continuous. And by Weierstrass and $f \in \mathcal{S}(\mathbb{R})$, we know $\sum_{n=-\infty}^{\infty} f(x+n)$ converges uniformly. By theorem 15.3.2 (Analysis) , F_1 is continuous. $\square$

$\underline{Property2:}$ F_1 is periodic with period 1.
pf. $F_1(x+1) = F_1(x)$ because $\phi : x+1 \mapsto x$ is a bij.. $\square$

To arrive another "period version" of f, by Fourier Analysis
Consider the discrete analogue of

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi,$$

then the integral is replaced by a sum

$$F_2(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x} = \sum_{n=-\infty}^{\infty} e^{2\pi i n x} \int_{-\infty}^{\infty} f(x) e^{-2\pi i n x} dx.$$

$\underline{Property1:}$ F_2 is continuous.
pf. Similar to Property1 of Way1, since $\hat{f} \in \mathcal{S}(\mathbb{R})$. $\square$

$\underline{Property2:}$ F_2 is periodic with period 1.
pf. Similar to Property2 of Way1. $\square$

The Poisson summation formula tells us that the F_1 and F_2 actually lead to the same function.

Theorem 3.1 (Poisson summation formula)
If $f \in \mathcal{S}(\mathbb{R})$, then

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

In particular, setting $x = 0$ we have

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

It means the Fourier coefficients of the periodization of f are given precisely by the values of the Fourier transform of f on the integers.

pf.
Theorem2.1 in Chapter 2 (Suppose that f is an integrable function on the circle with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(\theta_0) = 0$ whenever f is continuous at θ_0.) tells us that to prove the equality, we only need to show that **both sides have the same Fourier coefficients** (viewed as functions on the circle), for

Fourier transform is linear. Clearly, the m^{th} Fourier coefficient of the right-hand is $\hat{f}(m)$. For the left-hand side we have

$$\begin{aligned}\hat{F}_1(m) = \int_0^1 \left(\sum_{n=-\infty}^{\infty} f(x+n) \right) e^{-2\pi i m x}\, dx &\overset{*}{=} \sum_{n=-\infty}^{\infty} \int_0^1 f(x+n) e^{-2\pi i m x}\, dx \\ &= \sum_{n=-\infty}^{\infty} \int_n^{n+1} f(y) e^{-2\pi i m y}\, dy \\ &= \int_{-\infty}^{\infty} f(y) e^{-2\pi i m y}\, dy \\ &= \hat{f}(m) \\ &= \hat{F}_2(m)\end{aligned}$$

where the interchange of the sum and integral is permissible since f is rapidly decreasing. (By $Weierstrass$, $\sum_{n=-\infty}^{\infty} f(x+n)$ uniformly converges. And $f(x+n)$ is integrable on $[a,b]$. Through Analysis Thm 15.3.6 , we know that they can interchange) $\square$

The theorem extends to the case when we merely assume that both f and $\hat{f}$ are of **moderate decrease** (f is continuous and $\exists A > 0$ s.t. $|f(x)| \leq \frac{A}{1+x^2}, \forall x \in \mathbb{R}$). (Summation uniformly converges and function integrable??)

It turns out that **the operation of periodization** is important in a number of questions, even when the Poisson summation formula does not apply.

Ex. $f(x) = \frac{1}{x}, x \neq 0$, $\sum_{n=-\infty}^{\infty} \frac{1}{x+n} = \pi cot \pi x$, when $x \notin \mathbb{Z}$.

Ex. $f(x) = \frac{1}{x^2}, x \neq 0$, $\sum_{n=-\infty}^{\infty} \frac{1}{(x+n)^2} = \frac{\pi^2}{(sin \pi x)^2}$, when $x \notin \mathbb{Z}$.

1.3 Theta and zeta function

Def. We define the **theta function** $\vartheta(s)$ for $s > 0$ by

$$\vartheta(s) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 s}.$$

Note. $s > 0$ ensures the **absolute convergence** of the series. To apply the Poisson summation formula and ensure $\vartheta(s) \in \mathcal{S}(\mathbb{R})$.

Properties of theta function:
Theorem3.2 $\sqrt{s}\vartheta(\frac{1}{s}) = \vartheta(s)$ whenever $s > 0$.

pf. A simple application of the Poisson summation formula to $f(x) = e^{-\pi s x^2}$ and $\hat{f}(\xi) = \frac{1}{\sqrt{s}} e^{-\pi \frac{\xi^2}{s}}$. (Both in Schwarz space)

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} e^{-\pi s n^2} = \sum_{n=-\infty}^{\infty} \vartheta(n);$$

$$\sum_{n=-\infty}^{\infty} \hat{f}(\xi) = \sum_{n=-\infty}^{\infty} \sqrt{s} e^{-\pi \frac{\xi^2}{s}} = \sqrt{s} \sum_{n=-\infty}^{\infty} \vartheta(\frac{\xi}{s}).$$

□

The Theta function $\vartheta(s)$ also extends to **complex values** of s when Re(s) > 0, and the function is still valid then.
pf. Ensure the absolute convergence. □

The theta function is intimately connected with an important function in number theory, the **zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for Re(s) > 1.
Later we will see that this function carries essential information about the prime numbers (see Chapter 8).

It also turns out that ζ, ϑ and another important function Γ are related by the following identity:

$$\frac{1}{\pi^{\frac{s}{2}}} \Gamma(\frac{s}{2}) \zeta(s) = \frac{1}{2} \int_0^{\infty} t^{\frac{s}{2}-1} (\vartheta(t) - 1) dt,$$

which is valid for $s > 1$.

- $\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t}$ dt, Re(s) > 1.

pf.

$$\begin{aligned}
\frac{1}{2}\int_0^\infty t^{\frac{s}{2}-1}(\vartheta(t)-1)dt &= \frac{1}{2}\int_0^\infty t^{\frac{t}{2}-1}(\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}-1)dt \\
&= \frac{1}{2}\int_0^\infty t^{\frac{s}{2}-1}(\sum_{n=1}^{\infty} e^{-\pi n^2 t}+\sum_{n=-\infty}^{-1} e^{-\pi n^2 t})dt \\
&= \int_0^\infty t^{\frac{s}{2}-1}\sum_{n=1}^{\infty} e^{-\pi n^2 t}dt \\
&= \sum_{n=1}^{\infty}\int_0^\infty t^{\frac{s}{2}-1}e^{-\pi n^2 t}dt \\
&= \sum_{n=1}^{\infty}\int_0^\infty (\frac{t}{\pi n^2})^{\frac{s}{2}-1}e^{-t}d\frac{t}{\pi n^2} \\
&= \sum_{n=1}^{\infty}(\frac{1}{\pi n^2})^{\frac{s}{2}}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}dt.
\end{aligned}$$

And

$$\frac{1}{\pi^{\frac{s}{2}}}\Gamma(\frac{s}{2})\zeta(s) = \sum_{n=1}^{\infty}(\frac{1}{\pi n^2})^{\frac{s}{2}}\int_0^\infty t^{\frac{s}{2}-1}e^{-t}dt.$$

$\square$

Returning to the function ϑ, define the **generalization**

$$\Theta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2\tau}e^{2\pi i n z}$$

whenever $\mathrm{Im}(\tau) > 0$ and $z \in \mathbb{C}$. Taking $z = 0$ and $\tau = is$ we get $\Theta(z|\tau) = \vartheta(s)$. So the above equation can be expressed by

$$\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \frac{1}{2}\int_0^\infty [\vartheta(0;it)-1]\, t^{\frac{s}{2}}\frac{dt}{t}.$$

1.4 Heat kernels

Another application related to the Poisson summation formula and the theta function is the time-dependent heat equation on the circle (P118).
A solution to the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

subject to $u(x,0) = f(x)$, where f is periodic of period 1, was given by the previous chapter by

$$u(x,t) = (f * H_t)(x)$$

where $H_t(x)$ is **the heat kernel on the circle**, that is,

$$H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x}.$$

Note. $\Theta(x|4\pi i t) = H_t(x)$.

Also, recall that the **heat equation on** $\mathbb{R}$ gave rise to the heat kernel

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$$

where $\hat{\mathcal{H}}_t(\xi) = e^{-4\pi^2 \xi^2 t}$.

The fundamental relation between these two objects is an immediate consequence of the Poisson summation formula:

$\underline{Theorem 3.3}$ The heat kernel on the circle is the periodization of the heat kernel on the real line:

$$H_t(x) = \sum_{n=-\infty}^{\infty} \mathcal{H}_t(x+n).$$

pf. $H_t(x) = \sum_{n=-\infty}^{\infty} e^{-4\pi^2 n^2 t} e^{2\pi i n x} = \sum_{n=-\infty}^{\infty} \hat{\mathcal{H}}_t(n) e^{2\pi i n x} = \sum_{n=-\infty}^{\infty} \mathcal{H}_t(x+n)$. $\square$

And we also can prove that $H_t(x)$ is a good kernel on the circle:
$\underline{Corollary 3.4}$ The kernel $H_t(x)$ is a good kernel for $t \to 0$.
pf. We already observed that $\int_{|x|\leq \frac{1}{2}} H_t(x)dx = 1$. Now note that $H_t \geq 0$, which is immediate from the above formula since $\mathcal{H}_t \geq 0$.
We **claim** that when $|x| < \frac{1}{2}$,

$$H_t(x) = \mathcal{H}_t(x) + \varepsilon_t(x),$$

where the error satisfies $|\varepsilon_t(x)| \leq c_1 e^{-\frac{c_2}{t}}$ with $c_1, c_2 > 0$ and $0 < t \leq 1$.
To see this, note again that the formula in the theorem gives

$$H_t(x) = \mathcal{H}_t(x) + \sum_{|n|\geq 1} \mathcal{H}_t(x+n);$$

therefore, since $|x| \leq \frac{1}{2}$,

$$\varepsilon_t(x) = \frac{1}{\sqrt{4\pi t}} \sum_{|n|\geq 1} e^{-\frac{(x+n)^2}{4t}} \leq C \frac{1}{\sqrt{t}} \sum_{n\geq 1} e^{-\frac{cn^2}{t}}.$$

Note that $\frac{n^2}{t} \geq n^2$ and $\frac{n^2}{t} \geq \frac{1}{t}$ whenever $0 < t \leq 1$, so $e^{-c\frac{n^2}{t}} \leq e^{-\frac{c}{2t}} e^{-\frac{c}{2}n^2}$

$$|\varepsilon_t(x)| \leq C\frac{1}{\sqrt{t}}e^{-\frac{c}{2t}}\sum_{n\geq 1} e^{-\frac{c}{2}n^2} \leq c_1 e^{-\frac{c_2}{t}} \text{ (Since } \frac{1}{\sqrt{t}}e^{-\frac{c}{2}n^2} \in \mathcal{S}(\mathbb{R})).$$

As a result $\int_{|x|\leq\frac{1}{2}} |\varepsilon_t(x)|dx \to 0$ as $t \to 0$.
So $\int_{\eta<|x|\leq\frac{1}{2}} |H_t(x)|dx \to 0$ as $t \to 0$ because $\mathcal{H}_t$ does. □

1.5 Poisson kernels

In a similar manner to the discussion above about the heat kernels, we state the relation between the Poisson kernels (P55) for the disc and the upper half-plane where

$$P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|}e^{in\theta} = \frac{1-r^2}{1-2rcos\theta+r^2}(0 \leq r < 1)(By\ Lemma\ 5.5)$$

and

$$\mathcal{P}_y(x) = \frac{1}{\pi}\frac{y}{x^2+y^2}.$$

$\underline{Theorem 3.5}$ $P_r(2\pi x) = \sum_{n\in\mathbb{Z}} \mathcal{P}_y(x+n)$ where $r = e^{-2\pi y}$.
$pf.$ $P_r(2\pi x) = \sum_{n\in\mathbb{Z}} e^{-2\pi|n|y}e^{2\pi inx}$, while $\sum_{n\in\mathbb{Z}} \mathcal{P}_y(x+n) = \sum_{n\in\mathbb{Z}} e^{-2\pi|n|y}e^{2\pi inx} = P_r(2\pi x)$. □

2 5.4 The Heisenberg uncertainty principle

The mathematical thrust of the principle can be formulated in terms of a relation between a function and its Fourier transform. The basic underlying law, formulated in its vaguest and most general form, states that a function and its Fourier transform cannot both be essentially localized. Somewhat more precisely, if the "preponderance" of the mass of a function is concentrated in an interval of length L, then the preponderance of the mass of its Fourier transform cannot lie in an interval of length essentially smaller than L^{-1} (or cL^{-1}?) . The exact statement is as follows.

(Wikipedia) By Wikipedia, it can be described as the more accurately one property is measured, the less accurately the other property can be known. Furthermore, this property can be explained by Fourier transform.

$\underline{Theorem 4.1}$ Suppose $\psi \in \mathcal{S}(\mathbb{R})$ which satisfies the normalizing condition $\int_{-\infty}^{\infty} |\psi(x)|^2dx = 1$. Then

$$(\int_{-\infty}^{\infty} x^2|\psi(x)|^2dx)(\int_{-\infty}^{\infty} \xi^2|\hat{\psi}(\xi)|^2d\xi) \geq \frac{1}{16\pi^2}$$

and equality holds iff $\psi(x) = Ae^{-Bx^2}$ where $B > 0$ and $|A|^2 = \sqrt{\frac{2B}{\pi}}$.
In fact, we have

$$(\int_{-\infty}^{\infty}(x-x_0)^2|\psi(x)|^2dx)(\int_{-\infty}^{\infty}(\xi-\xi_0)^2|\hat{\psi}(\xi)|^2d\xi) \geq \frac{1}{16\pi^2}$$

for $\forall x_0, \xi_0 \in \mathbb{R}$.

pf.

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty}|\psi(x)|^2dx \\
&= x|\psi(x)|^2\Big|_{x=-\infty}^{x=\infty} - \int_{-\infty}^{\infty}x\frac{d}{dx}|\psi(x)|^2dx \\
&= -\int_{-\infty}^{\infty}x\frac{d}{dx}|\psi(x)|^2dx \ (Since\ \psi \in \mathcal{S}(\mathbb{R}),\ we\ have\ x|\psi(x)|^2\Big|_{x=-\infty}^{x=\infty} = 0.) \\
&= -\int_{-\infty}^{\infty}x\frac{d}{dx}\psi(x)\overline{\psi(x)}dx \\
&= -\int_{-\infty}^{\infty}(x\psi^{'}(x)\overline{\psi(x)} + x\overline{\psi^{'}(x)}\psi(x))dx \ (|\psi|^2 = \psi\overline{\psi}) \\
&\leq 2\int_{-\infty}^{\infty}|x||\psi^{'}(x)||\psi(x)|dx \\
&\leq 2(\int_{-\infty}^{\infty}|x|^2|\psi(x)|^2dx)^{\frac{1}{2}}(\int_{-\infty}^{\infty}|\psi^{'}(x)|^2dx)^{\frac{1}{2}} \ (By\ Cauchy-Schwarz\ inequality)
\end{aligned}$$

For the property of Fourier transform and the Plancherel formula (If $f \in \mathcal{S}(\mathbb{R})$, then $\parallel \hat{f} \parallel = \parallel f \parallel$.), we have

$$\int_{-\infty}^{\infty}|\psi^{'}(x)|^2dx = 4\pi^2\int_{-\infty}^{\infty}\xi^2|\hat{\psi}(\xi)|^2d\xi.$$

By *Prop.*1.2(*iv*) in Chapter 5, $|\hat{\psi'}(\xi)| = |2\pi i\xi\hat{\psi}(\xi)| = 2\pi\xi\hat{\psi}(\xi) = 2\pi\xi|\hat{\psi}(\xi)|$.
If equality hold, then we must also have equality where we applied the Cauchy-Schwarz inequality, and as a result we find that $\psi^{'}(x) = \beta x\psi(x)$ for some constant β. The solutions to this equation are $\psi(x) = Ae^{\beta\frac{x^2}{2}}$, where A is a constant.
Since we want ψ to be a Schwarz function, we must take $\beta = -2B < 0$, and since we impose the condition $\int_{-\infty}^{\infty}|\psi(x)|^2dx = 1$ we find that $|A|^2 = \sqrt{\frac{2B}{\pi}}$, as was to be shown (which is a normal distribution, $\mu = 0$ and $\sigma = \frac{1}{\sqrt{\beta}}$, thus $A = \sqrt{\frac{\beta}{2\pi}}$).
For the second inequality actually follows from the first by replacing $\psi(x)$ by

$e^{-2\pi ix\xi_0}\psi(x_0+x)$ and changing variables.

$$\begin{aligned}\int_{-\infty}^{\infty} x^2|\psi(x)|^2dx \Rightarrow \int_{-\infty}^{\infty} x^2|e^{-2\pi ix\xi_0}\psi(x_0+x)|^2dx \\ &= \int_{-\infty}^{\infty}(x-x_0)^2|e^{2\pi i(x_0-x)\xi_0}\psi(x)|^2dx \\ &= \int_{-\infty}^{\infty}(x-x_0)^2|\psi(x)|^2dx (\text{Since } \mathrm{e}^{2\pi\mathrm{i}(\mathrm{x}_0-\mathrm{x})\xi_0} \in \mathrm{K}_\sigma \text{ and Theorem.1.6})\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty}\xi^2|\hat{\psi}(\xi)|^2d\xi \Rightarrow \int_{-\infty}^{\infty}\xi^2|\hat{\psi}(\xi+\xi_0)|^2d\xi \\ &= \int_{-\infty}^{\infty}(\xi-\xi_0)^2|\hat{\psi}(\xi)|^2d\xi\end{aligned}$$

where

$$e^{-2\pi ix\xi_0}\hat{\psi}(x_0+x)(\xi) = \hat{\psi}(\xi+\xi_0)(\text{Prop.1.2 P136})$$

$\square$

2.1 *Application*: in the study of quantum mechanics.

It arose when one considered the extent to which one could simultaneously locate the **position** and **momentum** of a particle.

Position of the particle
Assuming we are dealing with an electron that travels along the real line, then according to the laws of physics, matters are governed by a **"state function"** , which we can assume to be in $\mathcal{S}(\mathbb{R})$, and which is normalized according to the requirement that

$$\int_{-\infty}^{\infty}|\psi(x)|^2dx = 1.$$

The position of the particle is then determined not as a definite point x; instead its probable location is given by the rules of quantum mechanics as follows: The probability that the particle is located in the interval (a,b) is $\int_a^b|\psi(x)|^2dx$. In addition to the **probability density** $|\psi(x)|^2dx$, there is the **expectation** of where the particle might be. This expectation is the best guess of the position of the particle, given its probability distribution determined by $|\psi(x)|^2dx$, and is the quantity defined by

$$\overline{x} = \int_{-\infty}^{\infty}x|\psi(x)|^2dx.$$

This is learned by our Probability course. (How to calculate expectation from discrete variables to continuous variables)

Variance is used to describe the uncertainty attached to our expectation. Having determined that the expected position of the particle is x, the resulting uncertainty is the quantity

$$\int_{-\infty}^{\infty}(x-\overline{x})^2|\psi(x)|^2dx.$$

Notice that if ψ is highly concentrated near $\overline{x}$, it means that there is a high probability that x is near $\overline{x}$, and so Var is small, because most of the contribution to the integral takes place for values of x near $\overline{x}$. Here we have a small uncertainty. On the other hand, if $\psi(x)$ is rather flat (that is, the probability distribution $|\psi(x)|^2dx$ is not very concentrated), then the integral **Var** is rather big, because large values of $(x-\overline{x})^2$ will come into play, and as a result the uncertainty is relatively large.
It is also worthwhile to observe that the expectation x is that choice for which the uncertainty $\int_{-\infty}^{\infty}(x-\overline{x})^2|\psi(x)|^2dx$ is the smallest. Indeed, if we try to minimize this quantity by equating to 0 its **derivative** with respect to $\overline{x}$, we find that $2\int_{-\infty}^{\infty}(x-\overline{x})|\psi(x)|^2dx = 0$, which gives the formula of probability density.
$(\overline{x} = \int_{-\infty}^{\infty}x|\psi(x)|^2dx = \int_{-\infty}^{\infty}\overline{x}|\psi(x)|^2dx = \overline{x}\int_{-\infty}^{\infty}|\psi(x)|^2dx = \overline{x})$

Momentum of the particle
The probability that the momentum ξ of the particle belongs to (a, b) is $\int_a^b|\hat{\psi}(\xi)|^2d\xi$ where $\hat{\psi}$ is the Fourier transform of ψ.
Useless......(By Wikipedia(**Position and momentum spaces**), in particular, if a function is given in position space, $f(r)$, then its Fourier transform obtains the function in momentum space, $\phi(p)$. Conversely, the inverse Fourier transform of a momentum space function is a position space function.)
Combining these two laws with Theorem 4.1 gives $\frac{1}{16\pi^2}$ as the lower bound for the product of the uncertainty of the position and the uncertainty of the momentum of a particle. So the more certain we are about the location of the particle, the less certain we can be about its momentum, and vice versa. However, we have simplified the statement of the two laws by rescaling to change the units of measurement. Actually, there enters a fundamental but small physical number called Planck's constant. When properly taken into account, the physical conclusion is

$$\text{(uncertainty of position)(uncertainty of momentum)} \geq \frac{\hbar}{16\pi^2}.$$

3 6.1 Preliminaries

The setting in this chapter will be $\mathbb{R}^d$, the vector space of all d-tuples of real numbers $(x_1, \cdots, x_d)$ with $x_i \in \mathbb{R}$.
Addition of vectors is component-wise, and so is multiplication by real scalars.

Given $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we define

$$|x| = (x_1^2 + \cdots + x_d^2)^{\frac{1}{2}},$$

so that $|x|$ is simply the length of the vector x in the usual Euclidean norm. In fact, we equip $\mathbb{R}^d$ with the standard inner product defined by

$$x \cdot y = x_1 y_1 + \cdots + x_d y_d,$$

so that $|x|^2 = x \cdot x$.
Given a d-tuple $\alpha = (\alpha_1, \cdots, \alpha_d)$ of non-negative integers (sometimes called a **multi-index**), the monomial x^α is defined by

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}.$$

Similarly, we define the differential operator $(\frac{\partial}{\partial x})^\alpha$ by

$$(\frac{\partial}{\partial x})^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} (\frac{\partial}{\partial x_2})^{\alpha_2} \cdots (\frac{\partial}{\partial x_d})^{\alpha_d} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}},$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_d$ is the order of the multi-index α.

3.1 Symmetries

Analysis in $\mathbb{R}^d$, and in particular the theory of the Fourier transform, is shaped by three important groups of symmetries of the underlying space:
(i) Translations;
(ii) Dilations;
(iii) Rotations.

In $\mathbb{R}$: We have seen that **translations** $x \mapsto x + h$, with $h \in \mathbb{R}^d$ fixed, and **dilations** $x \mapsto \delta x$, with $\delta > 0$, play an important role in the one-dimensional theory.
In $\mathbb{R}$, the only two rotations are the identity and multiplication by -1.

However, in $\mathbb{R}^d$ with $d \geq 2$ there are more rotations, and the understanding of the interaction between the Fourier transform and rotations leads to fruitful insights regarding spherical symmetries.

A **rotation** in $\mathbb{R}^d$ is a linear transformation R: $\mathbb{R}^d \to \mathbb{R}^d$ which preserves the inner product. In other words,

$$R(ax + by) = aR(x) + bR(y), \forall x, y \in \mathbb{R}^d, a, b \in \mathbb{R},$$

$$R(x) \cdot R(y) = x \cdot y, \forall x, y \in \mathbb{R}^d.$$

Equivalently, this last condition can be replaced by $|R(x)| = |x|$ for all $x \in \mathbb{R}^d$, or $R^t = R^{-1}$ where R^t and R^{-1} denote the transpose and inverse of R, respectively. In particular, we have $\det(R) = \pm 1$, where $\det(R)$ is the determinant of R. If $\det(R) = 1$ we say R is a **proper rotation**; otherwise, we say that R is an **improper rotation**.

*Ex*1. In $\mathbb{R}$, there are two rotations: the identity which is proper, and multiplication by -1 which is improper.

*Ex*2. The rotations in the plane $\mathbb{R}^2$ can be described in terms of complex numbers. We identify $\mathbb{R}^2$ with $\mathbb{C}$ by assigning the point (x, y) to the complex number $z = x + iy$. Under this identification, all proper rotations are of the form $z \mapsto ze^{i\varphi}$ for some $\varphi \in \mathbb{R}$, and all improper rotations are of the form $z \mapsto \overline{z}e^{i\varphi}$ for some $\varphi \in \mathbb{R}$.
See Exercise 1 for the argument leading to this result.

*Exe*1. Suppose that R is a rotation in the plane $\mathbb{R}^2$, and let

$$R = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

denote its matrix with respect to the standard basis vector $e_1 = (1, 0)$ and $e_2 = (0, 1)$.
(a) Write the conditions $R^t = R^{-1}$ and $\det(R) = \pm 1$ in terms of equations in a, b, c, d.
(b) Show that there exists $\varphi \in \mathbb{R}$ such that $a + ib = e^{i\varphi}$.
(c) Conclude that if R is proper, then it can be expressed as $z \mapsto ze^{i\varphi}$, and if R is improper, then it takes the form $z \mapsto \overline{z}e^{i\varphi}$, where $\overline{z} = x - iy$.
Ans: (a) $a = d = \cos\theta$ *and* $b = -c = \sin\theta$; *(b)* $e^{i\varphi} = \cos\varphi + i\sin\varphi$;
(c) $(x + iy)e^{i\varphi} = (x + iy)(\cos\varphi + i\sin\varphi) = x\cos\varphi - y\sin\varphi + i(x\sin\varphi + y\cos\varphi)$, $\det(R) = \det \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} = (\cos\varphi)^2 + (\sin\varphi)^2 = 1$; $(x - iy)e^{i\varphi} = (x - iy)(\cos\varphi + i\sin\varphi) = x\cos\varphi + y\sin\varphi + i(x\sin\varphi - y\cos\varphi)$, $\det(R) = \det \begin{pmatrix} \cos\varphi & \sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix} = -(\cos\varphi)^2 - (\sin\varphi)^2 = -1$.

*Ex*3. Euler gave the following very simple geometric description of rotations in $\mathbb{R}^3$. Given a proper rotation R, there exists a unit vector γ so that:
(i) R fixes γ, that is, $R(\gamma) = \gamma$.
(ii) If $\mathcal{P}$ denotes the plane passing through the origin and perpendicular to γ, then $R : \mathcal{P} \to \mathcal{P}$, and the restriction of R to $\mathcal{P}$ is a rotation in $\mathbb{R}^2$.
Geometrically, the vector γ gives the direction of the axis of rotation. A proof of this fact is given in Exercise 2. Finally, if R is improper, then $-R$ is proper (since in $\mathbb{R}^3$ $\det(-R) = -\det(R)$), so R is the composition of a proper rotation and a symmetry with respect to the origin.

*Exe*2. Suppose that $R : \mathbb{R}^3 \to \mathbb{R}^3$ is a proper rotation.

(a) Show that $p(t) = \det(R - tI)$ is a polynomial of degree 3, and prove that there exists $\gamma \in S^2$ (where S^2 denotes the unit sphere in $\mathbb{R}^3$) with

$$R(\gamma) = \gamma.$$

[Hint: Use the fact that $p(0) > 0$ to see that there is $\lambda > 0$ with $p(\lambda) = 0$. Then $R - \lambda I$ is singular, so its kernel is non-trivial.]
(b) If $\mathcal{P}$ denotes the plane perpendicular to γ and passing through the origin, show that

$$R : \mathcal{P} \to \mathcal{P},$$

and that this linear map is a rotation. $p(0) > 0$ to see that there is $\lambda > 0$ with $p(\lambda) = 0$. Then $R - \lambda I$ is singular, so its kernel is non-trivial.
(b) Unsolved maybe for $\det(R) = 1$?.

*Ex*4. Given two orthonormal bases $\{e_1, \cdots, e_d\}$ and $\{e_1^{'}, \cdots, e_d^{'}\}$ in $\mathbb{R}^d$, we can define a rotation R by letting $R(e_i) = e_i^{'}$ for $i = 1, \cdots, d$.
Conversely, if R is a rotation and $\{e_1, \cdots, e_d\}$ is an orthonormal basis, then $\{e_1^{'}, \cdots, e_d^{'}\}$, where $e_i^{'} = R(e_i)$, is another orthonormal basis.

3.2 Integration on $\mathbb{R}_d$

Since we shall be dealing with