GENERALIZED SPECTRA AND APPLICATIONS TO FINITE DISTRIBUTIVE LATTICES

WENDY LOWEN, MARK SIOEN, AND WOUTER VAN DEN HAUTE

Abstract. In our previous paper on frames of continuous functions, the classical adjunction between topological spaces and frames was generalized to a setup in which an arbitrary topological frame replaces the two element chain. The relevant composition of adjoints yields an endofunctor on topological spaces which in general fails to be idempotent. In this paper we prove a formula for iterations of this functor under certain conditions. We apply our result to the construction of finite free distributive lattices and Boolean algebras.

Keywords: free Boolean algebra, free distributive lattice, (topological) frame, spectrum
MSC2020: 06B25, 06B30, 06D05, 06D22, 06E05, 54H10, 54H12

Dedicated with gratitude to the memory of our dear friend Bernhard Banaschewski

1. Introduction

Algebraic logic dates back at least to the nineteenth century with the work "The laws of thought" by Boole. It took another century before lattice theory was developed as a mathematical subject in the work of Birkhoff among others. A particularly important subclass of lattices is formed by the distributive lattices, encompassing the Boolean algebras. Birkhoff's famous representation theorem can be phrased as a duality of categories between finite posets on the one hand and finite distributive lattices on the other hand [2]. This result can be obtained as a special instance of a more general (opposite) adjunction between topological spaces and frames [11]. In this adjunction a key role is played by the two element frame $\mathbf{2} = \{0, 1\}$ with $0 \leq 1$, which can be endowed with the Sierpinski topology for which $\{1\}$ is open. In this case, we will denote it by $\mathbb{S}$.

In [10] this adjunction was extended to a situation where $\mathbf{2}$ is replaced by a more general topological frame $\mathbb{F}$, yielding (opposite) adjoint functors

$$\mathcal{O}_{\mathbb{F}} : \mathsf{Top} \to \mathbb{F}/\mathsf{Frm} : X \longmapsto \mathsf{Top}(X, \mathbb{F})$$

and

$$\mathrm{Spec}_{\mathbb{F}} : \mathbb{F}/\mathsf{Frm} \to \mathsf{Top} : L \longmapsto \mathbb{F}/\mathsf{Frm}(L, F)$$

between the categories of topological spaces and of $\mathbb{F}$-frames respectively, as recalled in section 2. In case $\mathbb{F} = \mathbf{2}$, we have $\mathbf{2}\text{-}\mathsf{Frm} \cong \mathsf{Frm}$, the category of frames. In this case for a topological space X, we have $\mathcal{O}_{\mathbf{2}}(X) \cong \mathrm{open}(X)$, and for a frame L we have $\mathrm{Spec}_{\mathbf{2}}(L) \cong \mathrm{Spec}_{\wedge}(L)$, the $\wedge$-spectrum of L.

It was observed in [10] that, in contrast to the classical case $\mathbb{F} = \mathbf{2}$, in general the compostion

$$\mathrm{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}} : \mathsf{Top} \longrightarrow \mathsf{Top}$$

is not idempotent and does not yield an "$\mathbb{F}$-sobrification" of topological spaces.

This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 817762).

The main goal of the current paper is the investigation of this functor $\mathrm{Spec}_{\mathbb{F}}\mathcal{O}_{\mathbb{F}}$ restricted to sober exponential spaces X and where $\mathbb{F} = \mathsf{Top}(Y, \mathbb{S})$ with Y exponential and sober. In our main theorem 3.1 for such X and $\mathbb{F}$ we construct a homeomorphism

$$\mathrm{Spec}_{\mathbb{F}}(\mathcal{O}_{\mathbb{F}}(X)) \to \mathsf{Top}(Y, X). \tag{1}$$

In order to appreciate this result it is instructive to look at the case $\mathbb{F} = \mathbb{S} = \mathsf{Top}(\{1\}, \mathbb{S})$. On the left-hand side of (1), we obviously have $\mathrm{Spec}_{\mathbb{S}}(\mathcal{O}_{\mathbb{S}}(X)) \simeq X$ and on the right-hand side we immediately obtain $\mathsf{Top}(\{1\}, X) = X$.

For a more general $\mathbb{F} = \mathsf{Top}(Y, \mathbb{S})$ note that the right-hand side of (1) is considerably simpler than the left-hand side.

In section 4 we elaborate two applications of our main theorem. First we denote the three element chain by $\mathbf{3}$. Taking $\mathbb{F} = \mathbf{3} = \mathsf{Top}(\mathbb{S}, \mathbb{S})$ in (1), in Theorem 4.5 we prove that the iteration

$$(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S})$$

is homeomorphic to the free distributive lattice on n generators. Recall that the number of elements of this lattice is known as the n-th Dedekind number and that these numbers have been established up to $n = 8$, [5], [9].

Secondly we again consider $\mathbb{S}$ and denote $\diamond = \mathbf{2} \times \mathbf{2}$. Taking $\mathbb{F} = \diamond = \mathsf{Top}(A_2, \mathbb{S})$ in (1), with A_2 the discrete topological space on two elements, in Theorem 4.8 we prove that the iteration

$$(\mathrm{Spec}_{\diamond}\mathcal{O}_{\diamond})^n(\mathbb{S})$$

is homeomorphic to the free Boolean algebra on n generators.

Acknowledgement. The authors express their profound gratitude to Eva Colebunders for all the topology they have learned from her throughout the years. The first named author thanks her in particular for several illuminating conversations while working on this project, especially on the topic of exponential topological spaces, as well as for help with references.

2. Preliminaries

We recall the concept of a topological frame [10], which plays a central role in this paper as it allows to represent a topological space X by a frame of functions on X with values in such a topological frame.

For $(\mathbb{F}, \leq)$ a frame endowed with a topology $\mathcal{T}_{\mathbb{F}}$ we call $(\mathbb{F}, \leq, \mathcal{T}_{\mathbb{F}})$ a *topological frame* provided that the operations

$$\wedge : \mathbb{F} \times \mathbb{F} \to \mathbb{F} : (a, b) \mapsto a \wedge b$$

$$\sup_{i\in I} : \mathbb{F}^I \to \mathbb{F} : (a_i)_{i\in I} \mapsto \sup_{i\in I} a_i$$

are continuous. We will also simply write $\mathbb{F}$ to denote a topological frame.

Given $(\mathbb{F}_1, \leq_1, \mathcal{T}_1)$ and $(\mathbb{F}_2, \leq_2, \mathcal{T}_2)$ topological frames a map $f : \mathbb{F}_1 \to F_2$ is called a *topological frame morphism* if $f : (\mathbb{F}_1, \mathcal{T}_1) \to (\mathbb{F}_2, \mathcal{T}_2)$ is continuous and $f : (\mathbb{F}_1, \leq_1) \to (\mathbb{F}_2, \leq_2)$ is a frame homomorphism.

Let $\mathbb{F}$ be a topological frame. Let Top be the category of topological spaces with continuous maps and let Frm be the category of frames with frame homomorphisms. Let $\mathbb{F}/\mathsf{Frm}$ denote the comma category. Objects of $\mathbb{F}/\mathsf{Frm}$ are frame homomorphisms $\mathbb{F} \to L$ and are called $\mathbb{F}$-frames. The frame $\mathbb{F}$ is considered to be an $\mathbb{F}$-frame endowed with the identity. A morphism between $(L, \gamma_L : \mathbb{F} \to L)$ and $(L', \gamma'_L : \mathbb{F} \to L')$ in $\mathbb{F}/\mathsf{Frm}$ is a frame homomorphism $h : L \to L'$ such that

$$h\gamma_L = \gamma'_L.$$

Next we explain the functors

$$\mathcal{O}_{\mathbb{F}} : \mathsf{Top} \to \mathbb{F}/\mathsf{Frm} : X \longmapsto \mathsf{Top}(X, \mathbb{F})$$

and

$$\mathrm{Spec}_{\mathbb{F}} : \mathbb{F}/\mathsf{Frm} \to \mathsf{Top} : L \longmapsto \mathbb{F}/\mathsf{Frm}(L, F)$$

from [10].

For a topological space X, there is a natural frame homomorphism

$$\mathbb{F} \to \mathsf{Top}(X, \mathbb{F}) : a \mapsto c_a, \tag{2}$$

where $c_a : X \to \mathbb{F} : x \mapsto a$ is the constant function with value a. The $\mathbb{F}$-*function frame* $\mathcal{O}_{\mathbb{F}}(X)$ of X is the frame $\mathsf{Top}(X, \mathbb{F})$ (for the pointwise order) endowed with (2).

Let $(L, \gamma_L : \mathbb{F} \to L)$ be an $\mathbb{F}$-frame. Consider the set $\mathsf{Frm}_{\mathbb{F}}(L, \mathbb{F}) = (\mathbb{F}/\mathsf{Frm})(L, \mathbb{F})$ of frame homomorphisms $\psi : L \to \mathbb{F}$ with $\psi\gamma_L = 1_{\mathbb{F}}$ and the source of maps

$$(\mathrm{ev}_l : \mathsf{Frm}_{\mathbb{F}}(L, \mathbb{F}) \to \mathbb{F} : f \mapsto f(l))_{l \in L}.$$

The $\mathbb{F}$-*spectrum* of L is the set

$$\mathrm{Spec}_{\mathbb{F}}(L) = \mathsf{Frm}_{\mathbb{F}}(L, \mathbb{F}) \tag{3}$$

endowed with the initial topology for the source $(\mathrm{ev}_l)_{l \in L}$.
For more details on frames we refer the reader to [11], [12] or [6].

Next we recall some known facts on exponential objects in Top, [4], [8], [7], [6], [3]. Let X be an object in a category with finite products. Then X is called *exponential* if the functor $- \times X$ has a right adjoint, usually denoted by $(-)^X$.

Our main theorem deals with exponential objects in the category Top. In Top the right adjoint to $- \times X$ is denoted by $\mathsf{Top}(X, -)$.

Explicitely X exponential in Top means that for any topological space Z, the set of all continuous functions from X to Z underlies a canonical topological function space $\mathsf{Top}(X, Z)$ such that the evaluation map

$$\mathrm{ev} : \mathsf{Top}(X, Z) \times X \to Z : (f, x) \mapsto f(x) \tag{4}$$

is continuous and for every topological space Y and continuous map $f : Y \times X \to Z$ the map

$$f^* : Y \to \mathsf{Top}(X, Z) \tag{5}$$

is continuous, where

$$f^*(y) : X \to Z : x \mapsto f(x, y),$$

making the following diagram commute:

$$\begin{tikzcd} \mathsf{Top}(X, Z) \times X \arrow[r, "\mathrm{ev}"] & Z \\ Y \times X. \arrow[u, "f^* \times 1_X"] \arrow[ur, "f"'] & \end{tikzcd} \tag{6}$$

For $g \in \mathsf{Top}(Y, \mathsf{Top}(X, Z))$ we denote the composition with the evaluation map

$$\overline{g} = \mathrm{ev}(g \times 1_X). \tag{7}$$

Concretely this means

$$\overline{g}(y, x) = g(y)(x).$$

The correspondences between maps in (5) and (7) define a homeomorphism

$$\mathsf{Top}(Y, \mathsf{Top}(X, Z)) \simeq \mathsf{Top}(Y \times X, Z), \tag{8}$$

known as the first exponential law. The second exponential law applies the fact that $\mathsf{Top}(X,-)$, as a right adjoint functor, preserves products

$$\mathsf{Top}(X, \prod Z_i) \simeq \prod_i \mathsf{Top}(X, Z_i). \tag{9}$$

Following [6] we call a topogical space X *locally compact*, if for every point x of X and for every neighborhood of V of x there exists a compact neighborhood K of x such that $x \in K \subseteq V$.

Sober spaces are exponential if and only if they are locally compact, as was shown in [6], [7]. For locally compact spaces the canonical topological function space is given by the compact-open topology, see Proposition 1.1 in [3]. Hence we can conclude that for X, Z topological spaces, with X sober and exponential, the canonical topological function space $\mathsf{Top}(X, Z)$ is the set of continuous functions endowed with the compact-open topology, with subbasis given by sets of the form

$$M_{K,O} = \{f \in \mathsf{Top}(X, Y) \mid f(K) \subseteq O\}, \tag{10}$$

with $K \subseteq X$ compact and $O \subseteq Y$ open. For more information on categorical concepts we refer to [1].

Finally we agree to use the following notations. We will write $\simeq$ to denote homeomorphism between topological spaces, $\approx$ to denote isomorphism between frames and $\cong$ to denote simultaneous homeomorphism of spaces and isomorphism of frames (via the same morphism, whenever appropriate). In particular for topological frames $\cong$ means isomorphism of topological frames.

3. The main theorem

In this section, we will prove that, under certain conditions, and for the topological frame $\mathbb{F} = \mathsf{Top}(Y, \mathbb{S})$,

$$\mathrm{Spec}_{\mathbb{F}}(\mathcal{O}_{\mathbb{F}}(X)) \simeq \mathsf{Top}(Y, X).$$

Theorem 3.1. *Let X be a sober exponential topological space, $\mathbb{F}$ the topological frame $\mathbb{F} = \mathsf{Top}(Y, \mathbb{S})$ for some sober exponential space Y. Then $\mathrm{Spec}_{\mathbb{F}}(\mathcal{O}_{\mathbb{F}}(X))$ and $\mathsf{Top}(Y, X)$ are homeomorphic.*

Proof. In order to reach our goal, we first show the auxilliary claim that

$$\mathsf{Frm}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}) \simeq \mathsf{Top}(Y, X) \times \mathsf{Top}(Y, Y)$$

and then consider the appropriate embedding

$$\mathsf{Frm}_{\mathbb{F}}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}) \hookrightarrow \mathsf{Frm}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}).$$

First note that by the first exponential law (8)

$$\mathsf{Top}(X, \mathbb{F}) = \mathsf{Top}(X, \mathsf{Top}(Y, \mathbb{S})) \simeq \mathsf{Top}(X \times Y, \mathbb{S})$$

since X and Y are exponential in Top. Further since both spaces are endowed with compatible pointwise orders we actually have an isomorphism as topological frames

$$\mathsf{Top}(X, \mathsf{Top}(Y, \mathbb{S})) \cong \mathsf{Top}(X \times Y, \mathbb{S}).$$

Hence

$$\mathsf{Frm}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}) \simeq \mathsf{Frm}(\mathsf{Top}(X \times Y, \mathbb{S}), \mathsf{Top}(Y, \mathbb{S})), \tag{11}$$

where we endow both sides of the form $\mathsf{Frm}(L, \mathbb{F})$ with the initial topology for the source

$$(\mathrm{ev}_l : \mathsf{Frm}(L, \mathbb{F}) \to \mathbb{F} : f \mapsto f(l))_{l \in L}.$$

Since $X \times Y$, as a product of two sober spaces, is sober [6], the map

$$\mathcal{O} : \mathsf{Top}(Y, X \times Y) \to \mathsf{Frm}(\mathsf{Top}(X \times Y, \mathbb{S}), \mathsf{Top}(Y, \mathbb{S})) : f \mapsto \mathcal{O}(f)$$

is bijective, with $\mathcal{O}(f)(h) = hf$.

We now consider the composition $\overline{\mathcal{O}}$ of $\mathcal{O}$ and the relevant homeomorphism in (11). By definition, $\overline{\mathcal{O}}(f)$ makes the diagram

$$\begin{tikzcd}
\mathsf{Top}(X \times Y, \mathbb{S}) \arrow[r, "\mathcal{O}(f)"] & \mathsf{Top}(Y, \mathbb{S}) \arrow[d, "\mathrm{Id}"] \\
\mathsf{Top}(X, \mathbb{F}) \arrow[u, "\overline{\cdot}"] \arrow[r, "\overline{\mathcal{O}}(f)"] & \mathbb{F}
\end{tikzcd}$$

commute. Concretely we thus have

$$\overline{\mathcal{O}}(f)(g) = \mathcal{O}(f)(\overline{g}) = \overline{g}f \tag{12}$$

We now prove that the map $\overline{\mathcal{O}}$ is a homeomorphism.

Obviously, the sets of the form

$$N_{K,\{1\}} = \{h \in \mathsf{Top}(Y, \mathbb{S}) \mid h(K) = \{1\}\}$$

with K compact in Y form a subbasis for the topology on $\mathbb{F} = \mathsf{Top}(Y, \mathbb{S})$. Hence the pre-images $\mathrm{ev}_g^{-1}(N_{K,\{1\}})$, with $g \in \mathsf{Top}(X, \mathbb{F})$ constitute a subbasis on $\mathsf{Frm}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F})$.

The topology on $\mathsf{Top}(Y, X \times Y)$ is generated by the subbasis consisting of sets of the form

$$M_{K,O} = \{f \in \mathsf{Top}(Y, X \times Y) \mid f(K) \subseteq O\},$$

with K compact in Y and O open in $X \times Y$.

Using (6) and (5), consider $1_O \in \mathsf{Top}(X \times Y, \mathbb{S})$, for O open in $X \times Y$ and the corresponding map $1_O^* : X \to \mathbb{F}$. Then

$$\begin{array}{rcl}
\overline{\mathcal{O}}(M_{K,O}) & = & \{\overline{\mathcal{O}}(f) \mid f \in \mathsf{Top}(Y, X \times Y), f(K) \subseteq O\} \\
& = & \{\overline{\mathcal{O}}(f) \mid f \in \mathsf{Top}(Y, X \times Y), 1_O(f(K)) = \{1\}\} \\
& = & \{\overline{\mathcal{O}}(f) \mid f \in \mathsf{Top}(Y, X \times Y), \overline{\mathcal{O}}(f)(1_O^*)(K) = \{1\}\} \\
& = & \{\varphi \in \mathsf{Frm}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}) \mid \varphi(1_O^*)(K) = \{1\}\} \\
& = & \mathrm{ev}_{1_O^*}^{-1}(N_{K,\{1\}}).
\end{array}$$

Hence $\overline{\mathcal{O}}$ maps subbasic open sets to subbasic open sets and is therefore an open bijection.

To see that $\overline{\mathcal{O}}$ is continuous, consider the source $(\mathrm{ev}_g)_{g \in \mathsf{Top}(X,\mathbb{F})}$ defining the initial topology on $\mathsf{Frm}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F})$. We prove that $\mathrm{ev}_g \overline{\mathcal{O}}$ is continuous for every $g \in \mathsf{Top}(X, \mathbb{F})$. Observe that $\mathrm{ev}_g \overline{\mathcal{O}}$ coincides with the function $\mathsf{Top}(Y, X \times Y) \to \mathsf{Top}(Y, \mathbb{S}) : f \mapsto \overline{g}f$ which is obtained by considering the action of the right adjoint functor $\mathsf{Top}(Y, -)$ on $\overline{g} : X \times Y \to \mathbb{S}$.

Next, since Y is exponential, by the second exponential law (9), we have

$$\mathsf{Top}(Y, X \times Y) \simeq \mathsf{Top}(Y, X) \times \mathsf{Top}(Y, Y).$$

Consider the canonical embedding

$$i : \mathsf{Frm}_{\mathbb{F}}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}) \hookrightarrow \mathsf{Frm}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}) \simeq \mathsf{Top}(Y, X) \times \mathsf{Top}(Y, Y).$$

We will now determine the image $\mathrm{Im}(i)$.

Consider $(f_1, f_2) \in \mathsf{Top}(Y, X) \times \mathsf{Top}(Y, Y)$ and the corresponding $f \in \mathsf{Top}(Y, X \times Y)$ with $f(y) = (f_1(y), f_2(y))$. Then $(f_1, f_2) \in \mathrm{Im}(i)$ if and only if $\overline{\mathcal{O}}(f)$ is an $\mathbb{F}$-frame homomorphism. This means that for every $h \in \mathbb{F}$ the constant map $c_h \in \mathsf{Top}(X, \mathbb{F})$ satisfies

$$\overline{\mathcal{O}}(f)(c_h) = h.$$

By (12) and (7) we have

$$\overline{\mathcal{O}}(f)(c_h) = \overline{c_h}f = h\mathrm{pr}_2 f = hf_2,$$

for $\mathrm{pr}_2 : X \times Y \to Y$. Hence $(f_1, f_2) \in \mathrm{Im}(i)$ if and only if $h\mathrm{Id}_Y = hf_2$ for every $h \in \mathbb{F}$.

Since Y is sober it fulfills the T_0 separation property, whence this is equivalent to $f_2 = \mathrm{Id}_Y$.

Consequently the map

$$\mathrm{pr}_1 i : \mathrm{Spec}_{\mathbb{F}}(\mathcal{O}_{\mathbb{F}}(X)) = \mathsf{Frm}_{\mathbb{F}}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F}) \to \mathsf{Top}(Y, X)$$

is bijective, for $\mathrm{pr}_1 : \mathsf{Top}(Y, X) \times \mathsf{Top}(Y, Y) \to \mathsf{Top}(Y, X)$. The fact that it is also a homeomorphism easily follows from the diagram

$$\begin{tikzcd}
\mathsf{Top}(Y, X) \arrow[r, leftrightarrow] \arrow[d, hook] & \mathrm{Spec}_{\mathbb{F}}(\mathcal{O}_{\mathbb{F}}(X)) \arrow[d, hook] \\
\mathsf{Top}(Y, X) \times \mathsf{Top}(Y, Y) \arrow[r, leftrightarrow] & \mathsf{Frm}(\mathsf{Top}(X, \mathbb{F}), \mathbb{F})
\end{tikzcd}$$

and this concludes the proof. □

4. Applications to iterated spectra

In this section we elaborate two applications of our main theorem. These applications deal with finite distributive lattices with the Scott topology [6]. Note that in the finite case the Scott topology coincides with the Alexandroff topology for which the open sets are the upsets. Since all spaces used in this section are finite and satisfy the T_0 separation property, they are exponential and sober.

In order to apply our main theorem 3.1, the Scott topology and the compact open topology have to coincide on function spaces. This will follow from two more preliminary results.

Proposition 4.1. *Let X, Y be topological spaces with X finite. Then the compact-open topology on $\mathsf{Top}(X, Y)$ coincides with the initial topology for the source*

$$(\mathrm{ev}_x : \mathsf{Top}(Y, X) \to Y)_{x \in X}.$$

Proof. The compact-open topology has a subbasis consisting of the sets of the form $M_{K,O} = \{f \in \mathsf{Top}(X, Y) \mid f(K) \subseteq O\}$ with $K \subseteq X$ and $O \subseteq Y$ open.

A subbasis for the initial topology on the other hand, is given by sets of the form $\mathrm{ev}_x^{-1}(O) = \{f \in \mathsf{Top}(X, Y) \mid f(x) \in O\}$ with $x \in X$. Since X is finite,

$$M_{K,O} = \bigcap_{x \in K} M_{\{x\},O} = \bigcap_{x \in K} \mathrm{ev}_x^{-1}(O)$$

is a finite intersection and hence both topologies coincide. □

Proposition 4.2. *Let L, F be finite distributive lattices endowed with the Scott topology. Then $\mathsf{Top}(L, F) = \mathsf{Ord}(L, F)$ and the Scott topology and the initial topology on $\mathsf{Top}(L, F)$ determined by the source $(\mathrm{ev}_l : \mathsf{Top}(L, F) \to F_l)_{l \in L}$, coincide.*

Proof. That $\mathsf{Top}(L, F) = \mathsf{Ord}(L, F)$, the set of all order preserving maps $L \to F$, follows since a continuous map $f \in \mathsf{Top}(L, F)$ is always order-preserving and an order-preserving map $f \in \mathsf{Ord}(L, F)$ clearly preserves upsets.

Note that $\mathsf{Top}(L, F)$ is a subset of F^L and hence can be endowed with the trace of the product topology as can be compared to the previous theorem. $\mathsf{Ord}(L, F)$ is a lattice for the pointwise order which can be endowed with the Scott topology. As we deal with finite products, it is an easy consequence of Theorem II-4.13 in [6] that both topologies coincide. □

For the first application we consider the three element chain $\mathbf{3} = \mathsf{Top}(\mathbb{S}, \mathbb{S})$. In Theorem 4.5 below we prove that the iteration

$$(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S})$$

is homeomorphic to the free distributive lattice on n generators. Recall that the number of elements of this lattice is known as the n-th Dedekind number and that these numbers have been established up to $n = 8$, [5].

Before we can make the link between generalized spectra and free distributive lattices, we need an intermediate step through monotone Boolean functions.

Definition 4.3. *A monotone Boolean function of n variables is a non-decreasing map $\{0,1\}^n \to \{0,1\}$ where $\{0,1\}^n$ is endowed with the pointwise order. The set of monotone Boolean functions of n variables is denoted by M_n.*

Note that $M_n = \mathsf{Ord}(\mathbb{S}^n, \mathbb{S}) = \mathsf{Top}(\mathbb{S}^n, \mathbb{S})$. Moreover, note that

$$\mathsf{Top}(\mathbf{1}, \mathbf{3}) = \mathbf{3} \to \mathsf{Top}(\mathbb{S}, \mathbb{S}) : \begin{cases} 0 \mapsto (0,0) \\ 1 \mapsto (0,1) \\ 2 \mapsto (1,1) \end{cases}$$

constitutes an isomorphism of posets and hence of spaces, proving that $\mathbf{3} \cong \mathsf{Top}(\mathbb{S}, \mathbb{S})$. The following result can be found in, e.g., [2].

Theorem 4.4. *The lattice of monotone Boolean functions of n variables is isomorphic to the free distributive lattice on n generators.*

Theorem 4.5. $(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S})$ *and M_n are homeomorphic for $n \geq 1$.*

Proof. We will give a proof by induction. Applying Theorem 3.1 with $X = \mathbb{S}$, $\mathbb{F} = \mathbf{3} = \mathsf{Top}(\mathbb{S}, \mathbb{S})$ we can see that this is true for $n = 1$.

So assume that $(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^{n-1}(\mathbb{S})$ and $\mathsf{Top}(\mathbb{S}^{n-1}, \mathbb{S}) = M_{n-1}$ are homeomorphic for some $n \geq 2$. Applying Theorem 3.1 with $X = \mathsf{Top}(\mathbb{S}^{n-1}, \mathbb{S})$ and $\mathbb{F} = \mathbf{3} = \mathsf{Top}(\mathbb{S}, \mathbb{S})$, and the first exponential law (8), we get

$$\begin{aligned} (\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S}) &= \mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}}((\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^{n-1}(\mathbb{S})) \\ &\simeq \mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}}(\mathsf{Top}(\mathbb{S}^{n-1}, \mathbb{S})) \\ &\simeq \mathsf{Top}(\mathbb{S}, \mathsf{Top}(\mathbb{S}^{n-1}, \mathbb{S})) \\ &\simeq \mathsf{Top}(\mathbb{S}^n, \mathbb{S}) \simeq M_n, \end{aligned}$$

which finishes the proof. □

Since all finite lattices in this theorem were endowed with the Scott topology, we get the following result.

Corollary 4.6. *M_n and $(\mathrm{Spec}_{\mathbf{3}}\mathcal{O}_{\mathbf{3}})^n(\mathbb{S})$ are isomorphic lattices for $n \geq 1$.*

For a second application let A_n denote the antichain on n elements. We start again from $\mathbb{S}$ and denote $\diamond = \mathbb{S} \times \mathbb{S} = \mathsf{Top}(A_2, \mathbb{S})$. Taking $\mathbb{F} = \diamond$ in Theorem 3.1 we prove that the iteration

$$(\mathrm{Spec}_{\diamond}\mathcal{O}_{\diamond})^n(\mathbb{S})$$

is homeomorphic to B_n, the free Boolean algebra on n generators.

Lemma 4.7. *For $n \geq 1$,* $\mathsf{Top}(A_{2^{n-1}}, \diamond) \cong \mathsf{Ord}(A_{2^{n-1}}, \diamond) \cong B_n$.

Proof. Since the discrete and the Scott topology on A_n coincide it follows that $\mathsf{Ord}(A_{2^{n-1}}, \diamond) = \diamond^{2^{n-1}}$. □

Theorem 4.8. $(\mathrm{Spec}_{\diamond}\mathcal{O}_{\diamond})^n(\mathbb{S})$ *and B_n are homeomorphic for $n \geq 0$.*

Proof. For $n = 0$, this is trivial. By Lemma 4.7, we already know that $\mathsf{Top}(A_{2^{n-1}}, \diamond)$ and B_n are homeomorphic for any $n \geq 1$. We will now prove by induction that $(\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^n(\mathbb{S})$ and $\mathsf{Top}(A_{2^{n-1}}, \diamond)$ are homeomorphic for $n \geq 1$.

For $n = 1$ we take $X = \mathbb{S}$, $\mathbb{F} = \diamond = \mathsf{Top}(A_2, \mathbb{S})$ in 3.1. It follows that

$$\mathrm{Spec}_\diamond \mathcal{O}_\diamond(\mathbb{S}) = \mathsf{Top}(A_{2^1}, \mathbb{S}) = \diamond = \mathsf{Top}(A_{2^0}, \diamond).$$

Next assume that $(\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^{n-1}(\mathbb{S})$ and $\mathsf{Top}(A_{2^{n-2}}, \diamond) = B_{n-1}$ are homeomorphic for some $n \geq 2$. Applying Theorem 3.1 with $X = \mathsf{Top}(A_{2^{n-2}}, \diamond)$ and $\mathbb{F} = \diamond = \mathsf{Top}(A_2, \mathbb{S})$, Lemma 4.7 and the first exponential law (8),

$$\begin{aligned}
(\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^n(\mathbb{S}) &= \mathrm{Spec}_\diamond \mathcal{O}_\diamond((\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^{n-1}(\mathbb{S})) \\
&\simeq \mathrm{Spec}_\diamond \mathcal{O}_\diamond(\mathsf{Top}(A_{2^{n-2}}, \diamond)) \\
&\simeq \mathsf{Top}(A_2, \mathsf{Top}(A_{2^{n-2}}, \diamond)) \\
&\simeq \mathsf{Top}(A_{2^{n-1}}, \diamond) \simeq B_n,
\end{aligned}$$

which is what we wanted to show. □

Once again, since all finite lattices in this theorem were endowed with the Scott topology, we get the following result.

Corollary 4.9. *B_n and $(\mathrm{Spec}_\diamond \mathcal{O}_\diamond)^n(\mathbb{S})$ are isomorphic lattices for $n \geq 0$.*

References

[1] J. Adámek, H. Herrlich, G.E. Strecker, Abstract and Concrete Categories. The Joy of Cats, In: Pure and Applied Mathematics, John Wiley and Sons, Inc., New York, 1990.

[2] G. Birkhoff, Lattice Theory, American Math. Soc. Colloquium Publ, Providence, Rhode Island, 1967.

[3] E. Colebunders, G. Richter, An Elementary Approach to Exponential Spaces, Applied Categor. Struct. 9 (2001), 303-310.

[4] B.J. Day, G.M. Kelly, On topological quotient maps preserved by pullbacks or products, Proc. Camb. Phil. Soc. 67, (1970), 553-558.

[5] R. Fidytek, A.W. Mostowski, R. Somla, A. Szepietowski, Algorithms counting monotone Boolean function, Information Processing Letters 79 (2001), 203-209.

[6] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M.W. Mislove and D.S. Scott, Continuous lattices and domains, Cambridge University Press, 2003.

[7] K.H. Hofmann, M.W. Mislove, The spectral theory of distributive continuous lattices, Trans. Amer. Math. Soc. 246 (1978), 285-310.

[8] J. Isbell, General function spaces, products and continuous lattices, Math. Proc. Camb. Phil. Soc. 100 (1986), 193-205.

[9] D. Kleitman, On Dedekind's Problem: The number of monotone Boolean functions, Proc. Amer. Math. Soc. 21(3) (1969), 677-682.

[10] W. Lowen, M. Sioen, W. Van Den Haute, Frames of continuous functions, Topology Appl., 27 (2020), 106974.

[11] J. Picado, A. Pultr, Frames and Locales. Topology Without Points, Frontiers in Mathematics, Birkhäuser/Springer, Basel AG, Basel, 2012.

[12] A. Pultr, Frames, in: Handbook of Algebra, vol. 3, Elsevier/North-Holland, Amsterdam, 2003, pp. 791–857.

(Wendy Lowen) Universiteit Antwerpen, Departement Wiskunde, Middelheimcampus, Middelheimlaan 1, 2020 Antwerp, Belgium

Email address: wendy.lowen@uantwerpen.be

(Mark Sioen – Corresponding author) Vrije Universiteit Brussel, Vakgroep Wiskunde, Pleinlaan 2, 1050 Brussels, Belgium

Email address: Mark.Sioen@vub.be

(Wouter Van Den Haute) Universiteit Antwerpen, Departement Wiskunde, Middelheimcampus, Middelheimlaan 1, 2020 Antwerp, Belgium

Vrije Universiteit Brussel, Vakgroep Wiskunde, Pleinlaan 2, 1050 Brussels, Belgium

Email address: Wouter.Van.Den.Haute@vub.be