A FIELD GUIDE TO HYPERBOLIC SPACE

An Exploration of the Intersection
of Higher Geometry and
Feminine Handicraft

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fig.

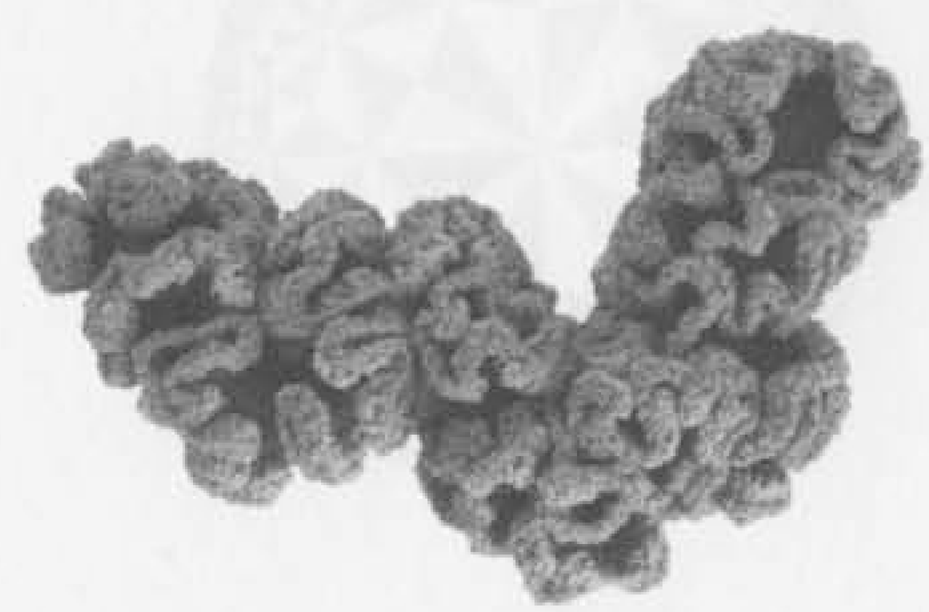
An Excess of Surface

We have built a world of rectilinearity.

The rooms we inhabit, the skyscrapers we work in, the grid-like arrangement of our streets and the freeways we cruise on our daily commute speak to us in straight lines.

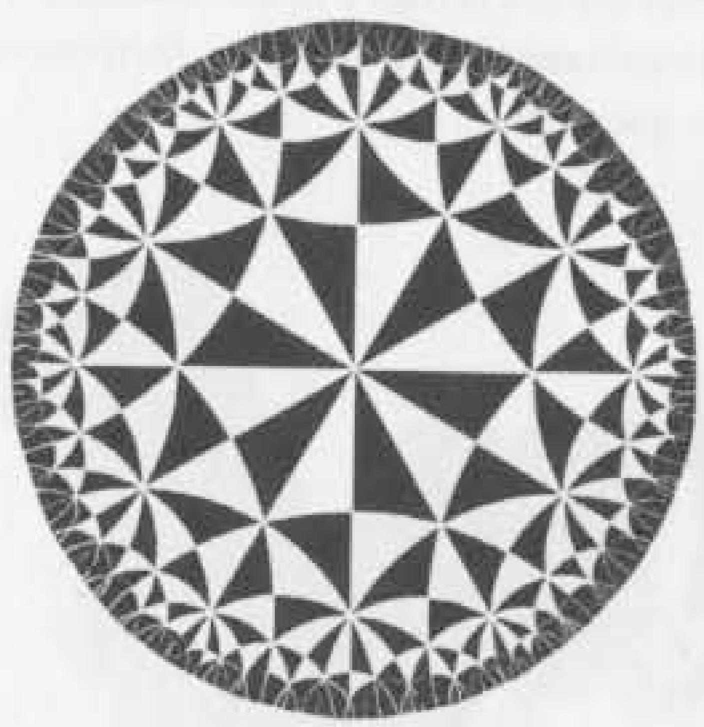
Yet outside our boxes the natural world teems with swooping, curling and crenellated forms, from the fluted surfaces of lettuces and fungi, to the frilled skirts of nudibranches and the animal undulations of sea slugs and anemonies.

We have learned to play by Euclidean rules because two thousand years of geometrical training have engraved the grid in our minds. But in the early nineteenth century mathematicians became aware of a space in which lines cavorted in aberrant formations, suggesting the existence of a new geometry.



To all at the time *hyperbolic space* seemed pathological, for it contravened the dictates of Euclid, overthrowing millenia of mathematical wisdom and offending common sense. "I fear the howl of the Boetians if I make my ideas known," wrote the immortal Carl Friedrich Gauss, who pioneered the topography of this polychrome realm.

A century later the Dutch artist M.C. Escher propelled hyperbolic space into the cultural zeitgeist with his "Circle Limit" series of etchings, tessellating birds and fishes or angels and demons to demonstrate in graphic play the superabundant structure hidden within this fantastical geometry.



Characterized by an almost organic excess, hyperbolic space resembles nothing so much as a sea creature.

Indeed, eons before the dawning of mathematical awareness, nature had exploited this verboten form, realizing its potential throughout the vegetable and marine kingdoms.

The human discovery of hyperbolic space initiated the formal field of non-Euclidean geometry and opened men's eyes to the possibility that the cosmos itself may have other options than the Cartesian box of canonical scientific faith.

Though it had long been thought that the space of our universe must *ipso facto* conform to Euclid's ideals, data coming from telescopic studies of the early universe now suggests that the cosmological whole may embody a hyperbolic form.

At the heart of our inquiry is the concept of straightness: What exactly is a straight line, and how do such objects relate to one another?

Though seemingly obvious, straightness turns out to be a subtle and surprisingly plastic concept.

To understand what is at stake here we must go back to Euclid and the original axioms of planar geometry. Long regarded as the model of intellectual rigor, Euclidean geometry is based on five supposedly self-evident axioms.

The first three are mundane enough, defining a line segment, an extended straight line, and a circle.

The fourth also seems uncontroversial and is usually interpreted to mean that all right angles are equal—a proposition necessary to ensure that the space we are working in is essentially the same everywhere. The property of spatial homogeneity is the defining quality of a *geometry*—for mathematically speaking there are wilder and more unruly realms.

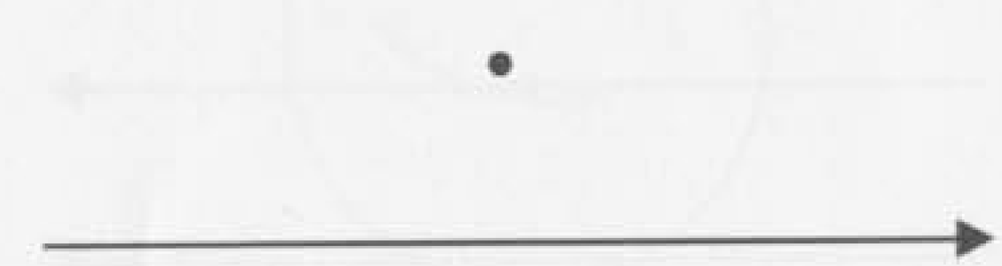
Euclid's fifth postulate also sounds eminently reasonable: it defines the conditions for parallel lines.

But mathematicians have always sensed that this apparently sensible proposition needed further investigation.

There are several ways of describing this fifth, troublesome axiom, also known as the *parallel postulate*.

Euclid's own method seems strange to modern eyes and mathematicians today prefer to use a construction popularized by the Scotsman John Playfair in the late nineteenth century.

In Playfair's description we may understand parallel lines in the following way: Imagine I draw a line, and then define a point outside that line.

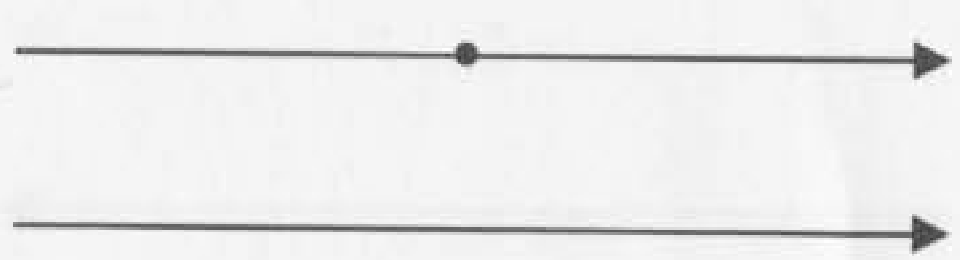


Now imagine I wish to draw other lines through the point. What is the result?

Euclid's fifth axiom says that there is no more than one line I can draw through the point that will never meet the original line.

All other lines will slant with respect to this and eventually intersect it. We call the non-intersecting lines *parallels* and denote them by arrows.

As stated above, the parallel postulate seems intuitively correct. But it is so much more complicated than Euclid's other axioms and from the beginning mathematicians felt haunted by the need for a less complex articulation. If the proposition really is true, they felt, then ought it not be provable from the other, simpler, axioms?



That there is an issue at all here is suggested by the example of a sphere, whose surface forms a geometry different to that of the plane.

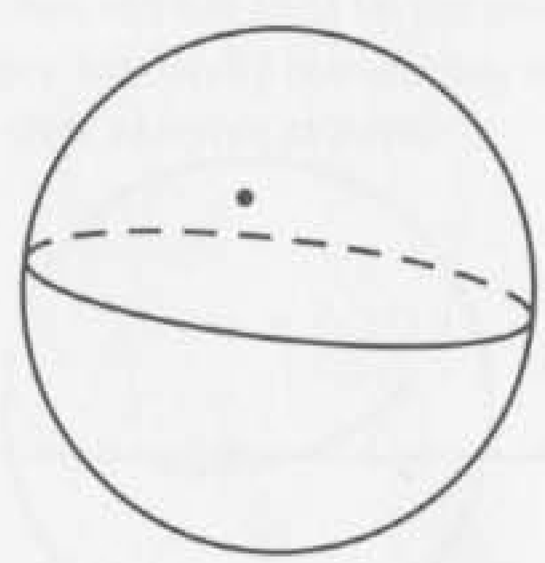
Again we may ask a question about the behavior of straight lines on this surface.

Suppose again that we draw a straight line and a point outside this line, both on the spherical surface. What happens now when we try to draw other lines through the point?

Immediately we are faced with a question: What does it mean to talk about straight lines on a curved surface?

Mathematically, a straight line may be generalized to the concept of a *geodesic*, a term that defines the shortest path between two points.

On a flat plane like a sheet of paper, the shortest distance is a path with no swerves or deviations, and likewise, on the surface of a sphere, we are looking for the minimal route.



Here we find that the shortest distance is always along a *great circle,* which divides the sphere into two halves like the earth's Equator. Great circles are the largest enclosings that may be drawn on any sphere.

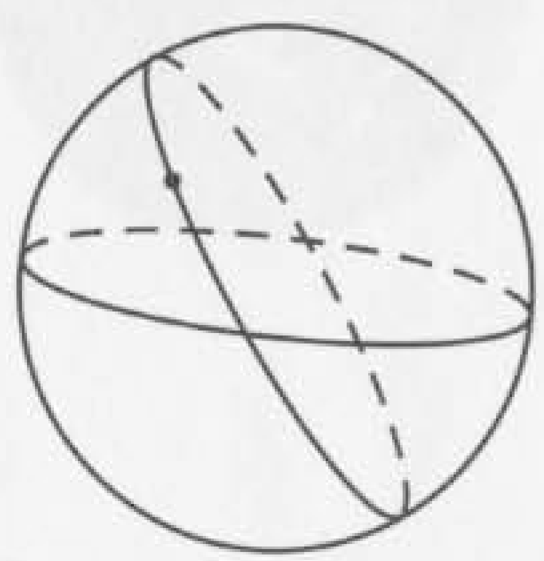
Airlines use such geodesics when charting the paths of international flights, which often look curved on a flat map but are "straight" in relation to the globe itself.

With respect to a sphere we notice that the geodesics are not infinite as they are on a plane, but finite—inevitably connecting back up on themselves.

Returning now to our initial question, we ask about the relationship between our original straight line and others we may draw through an external point.

In this case, any straight line through a point is also, by definition, another great circle, and all great circles intersect.

Thus on the surface of a sphere, there are *no* straight lines through a point that do not meet the original line. Whereas on the plane there is always one non-intersecting straight line, now we have a geometry in which all lines meet.



Euclid's postulate had stated that there can never be *more than one* line through a point that does not meet an original line. On a sphere there are none, so the postulate holds.

How do we know there isn't some other surface in which there may be two or more parallels?

The idea that one might not be the limit struck terror into mathematicians hearts, offending rational sensibilities and evoking a sense of moral outrage.

For two thousand years they sought to prove that such an option was impossible.

Through monumental effort they tried to demonstrate that if the parallel postulate was violated and more than one parallel was allowed then logical chaos would ensue.

What they discovered during this process was a host of bizarre effects—but no outright contradictions.

In the eighteenth century, the Jesuit priest Girolamo Sacchieri devoted his life to the problem of parallels but went to his Maker a failure in his own eyes, unable to demonstrate after Sisyphean effort a single logical disjunction.

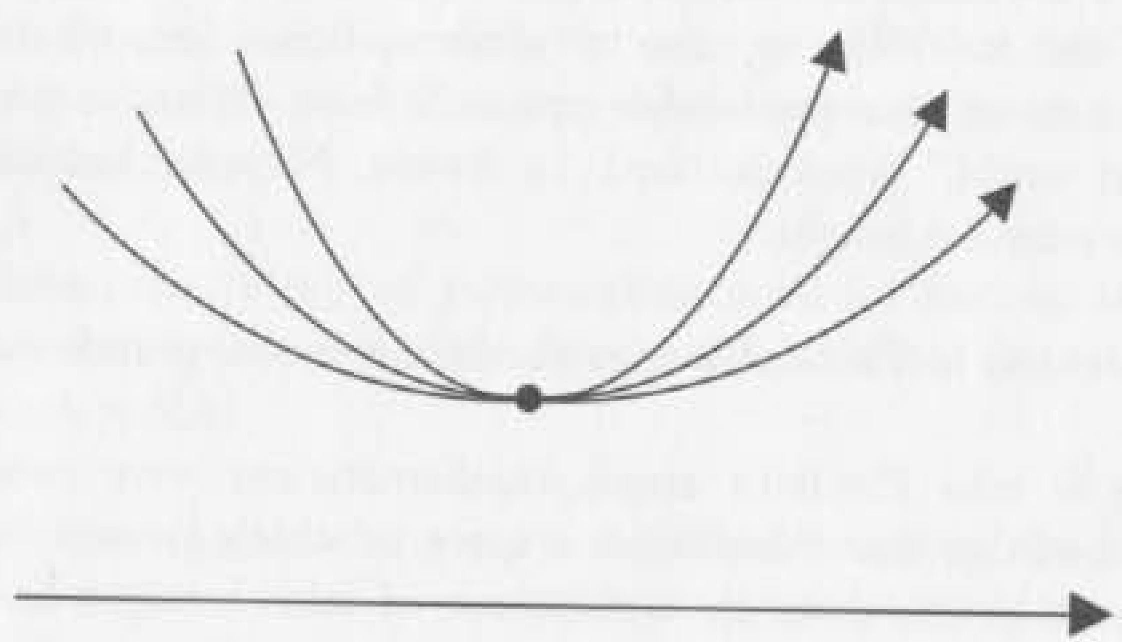
Even Gauss, the prince of mathematicians, could find no contradictions—fearing the howl of the Boetians, he refrained from making his research public.

Finally in the nineteenth century, the effort to prove the parallel postulate exhausted itself, as mathematicians accepted the mounting evidence for the existence of a geometry based upon its negation.

In 1823 the Hungarian mathematician Janos Bolyai wrote ecstatically to his father Wolfgang, also a mathematician, announcing his explorations of this improbable space. "I have created a new and different world," Janos declared. In Russia, Nickolai Lobachevsky came to a similar insight.

An alternative to Euclid, however disturbing, was now undeniable.

To put it into Playfair's terms, mathematicians were compelled to acknowledge that there exists a space in which given a line and an external point, there are a multitude of other straight lines that intersect with the point, yet never meet the original line.



Instead of there being just one parallel, there are many.

Indeed, there are infinitely many.

Bizarre though it may seem, this situation gives rise to a consistent geometry, what came to be called, in homage to its abundant excess, the *hyperbolic* plane.

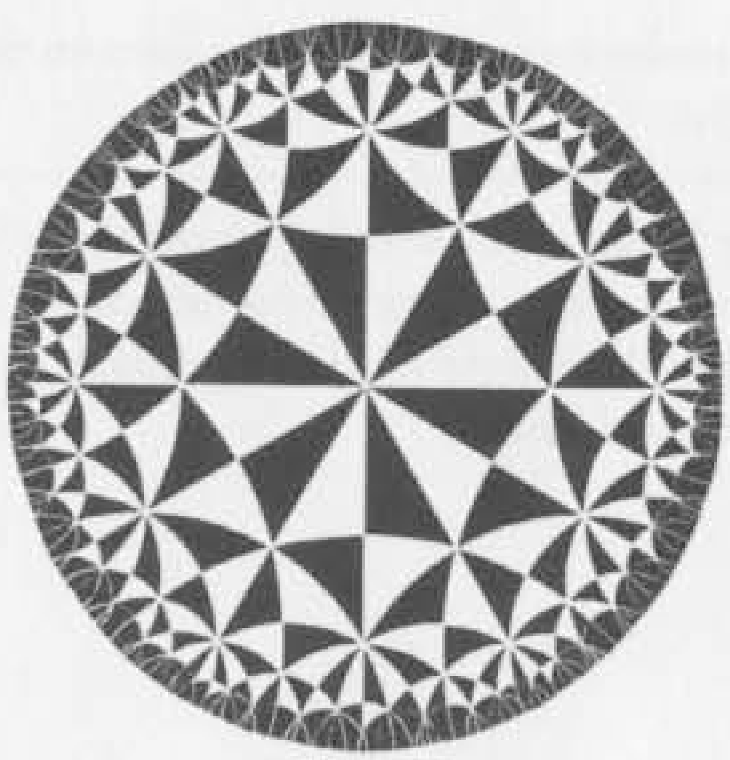
At this point the reader may object that the lines on the opposite page do not look straight.

But that is merely because we are trying to see them from our limited Euclidean perspective. From the point of view of someone within the hyperbolic space, all these lines would be perfectly straight and none would meet the original line.

It is one thing however to know that something is logically possible, it is quite another to understand it.

Like the blind man and the elephant, hyperbolic space appears in different guises depending on how we approach it.

One way of visualizing this enigmatic space was discovered at the end of the nineteenth century by the French mathematician Henri Poincaré. In the Poincaré disc model the entire hyperbolic plane is depicted inside a circular disc.



In reality, the hyperbolic plane is infinitely large—like the Euclidean plane, it goes on forever.

But in order for us to represent it within our Euclidean framework we have to make some compromises. The Poincaré compromise is to represent angles truly while distorting scale.

Despite appearances in the diagram opposite, all the sides of all the triangular shaped areas are equal in length. Though they appear to be decreasing in size as we move towards the perimeter, within the space itself the vertices of the triangles are equidistant and the boundary of the circle is infinitely far away.

In his book *Science and Hypothesis* (1901), Poincaré wrote of his model as an imaginary universe. To us, observers of this bubble world, the inhabitants of the disc appear to shrink as they approach the circular boundary—they, however, see no such effect. As far as they are concerned, they live in an infinite and non-diminishing space.

Only we, who must view them from a Euclidean framework, see their proportions fading away to infinitessimal nothingness.

The Poincaré disc model of hyperbolic space has entered the cultural lexicon through the work of the Dutch artist M.C. Escher, who was introduced to the concept by the geometer Donald Coxeter.

In his Circle Limit series of drawings, Escher explored the endless symmetries inherent in the hyperbolic plane: in Circle Limit III, red, green, blue and yellow fish tessellate their world in a symphony of triangles and squares.

In Circle Limit IV angels and demons disport themselves in a hyperbolic trinity, fluttering out from a central point to fill the space with hexagons and octagons.

In the playfulness of these images lies an elegant lesson: the excess of parallels in hyperbolic space opens up a richer field for the tessellating spirit, and the hyperbolic plane can be tiled in an almost infinite variety of ways.

At the same time that Escher was propelled by the formalities of geometry, he was also inspired in these explorations by a visit to the Alhambra Palace in Spain, that apotheosis of the Arab world's unparalleled tiling tradition.

If, as the Moors believed, repeated patterns connote the divine, we might conclude that Heaven itself would be a hyperbolic space.

Yet for all its evident beauty and power, the Poincaré disc model is essentially an abstract construct. It obscures at the same time that it reveals, for we do not get a sense here of what it would feel like to be *in* hyperbolic space.

By restricting ourselves to a Euclidean perspective we lose the visceral sense of hyperbolic being.

Can we make a model of hyperbolic space that retains this physical sense, much as we have a model of spherical space?

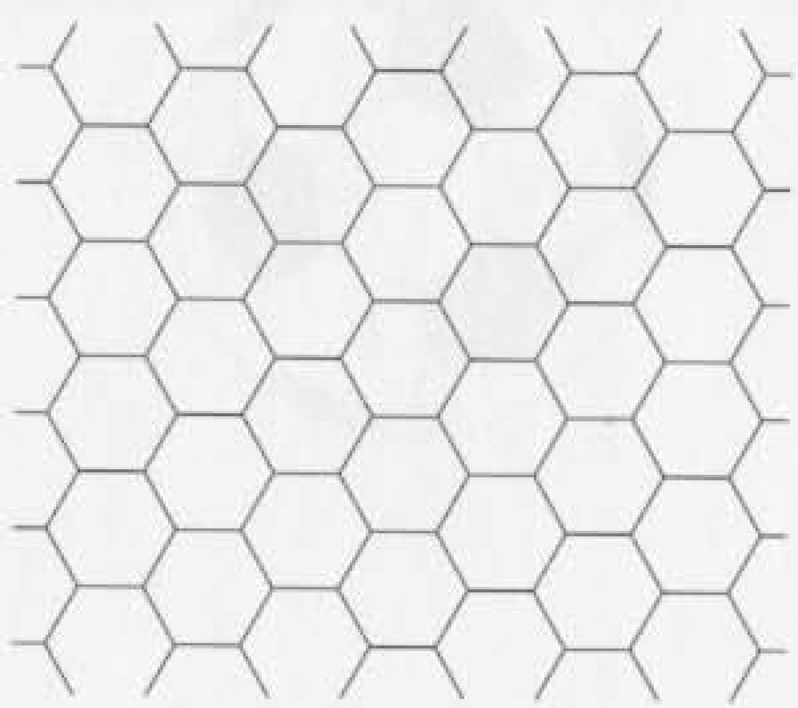
For a long time mathematicians did not believe that such a thing was possible.

Yet it turns out there is a way of representing hyperbolic space that gives us a visceral sense of at least some of its properties: the so-called hyperbolic soccer ball model, discovered by a young American mathematics teacher named Keith Henderson.

Think first of a regular soccer ball—it is made up of hexagons and pentagons, with a series of white hexagons surrounding a number of black pentagons.

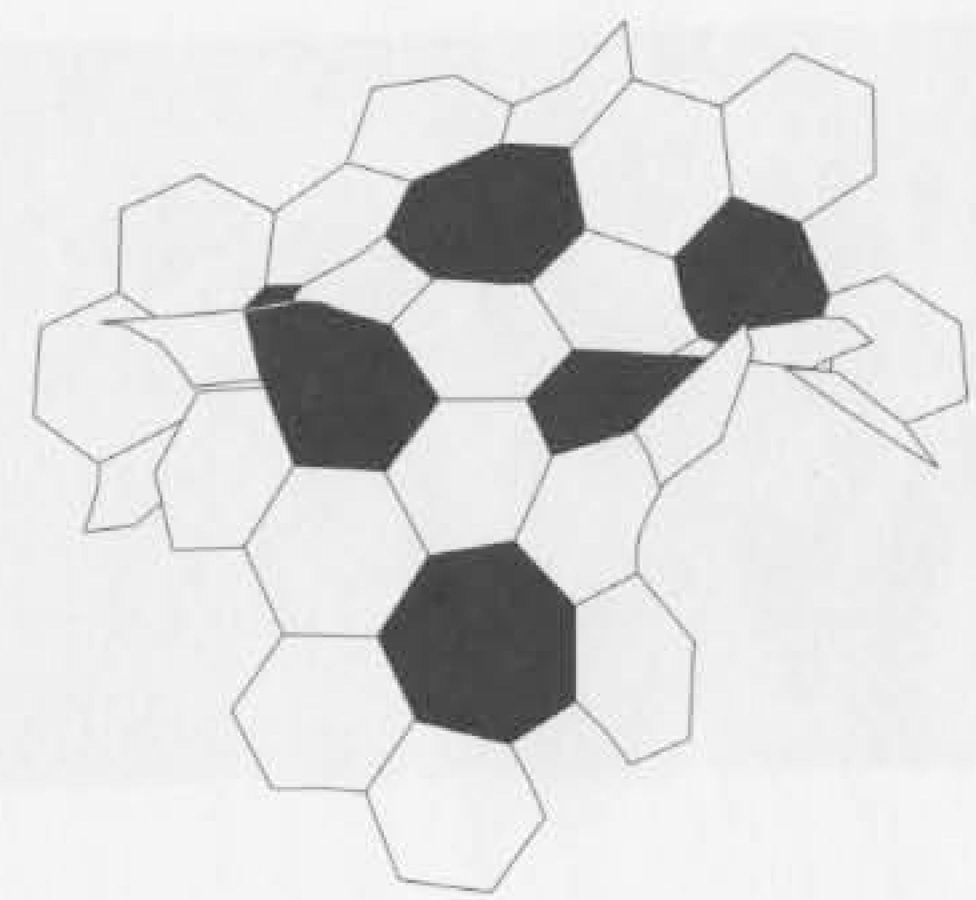
Now think of a Euclidean plane. Here we can tile the surface with hexagons alone in the classic beehive pattern. On the plane, every hexagon—which has six sides—is surrounded by six others that neatly fit together to exactly fill the space.

To make a soccer ball, we replace some of the hexagons with pentagons—which only have five sides—thereby causing the now fewer number of hexagons to close up towards one another and wrap into a sphere.



In the hyperbolic version of this model we make the opposite move. Rather than replacing hexagons with pentagons, we replace them with heptagons—which have seven sides.

Now, instead of closing up, the surface opens out, for the heptagons add to, rather than subtract from the space, resulting in an excess of surface.



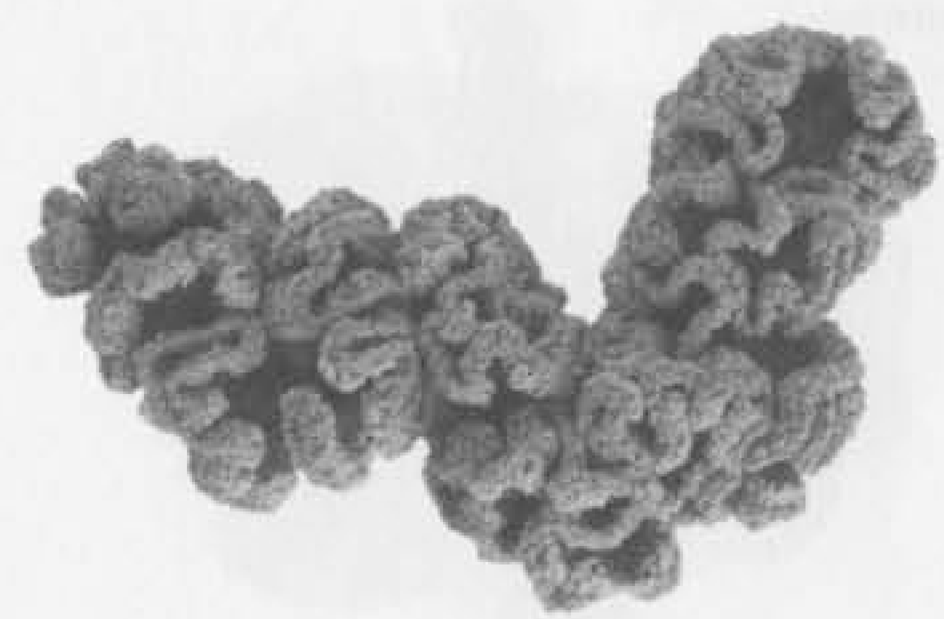
The effect is similar to what we see in mushrooms and fungi, lettuce leaves and kelps, wherein the vegetable surface expands outward from a modest beginning generating a ruffled effect.

Mathematicians now understand that lettuces and kelps are natural examples of hyperbolic geometry, which is also found in the anatomical frills of sea slugs, flatworms and nudibranches.

If nature can do it, then why not man?

Or perhaps woman?

In 1997 Latvian mathematician Daina Taimina finally worked out how to make a physical model of hyperbolic space that allows us to feel, and to tactilely explore the properties of this unique geometry. The method she used was crochet.



Dr. Taimina's inspiration was based on a suggestion that had been put forward in the 1970's by the American geometer William Thurston.

Thurston noted that one of the qualities of hyperbolic space is that as you move away from any point the space around it expands exponentially. Based on this insight, he designed a paper model made of thin crescent-shaped annuli taped together.

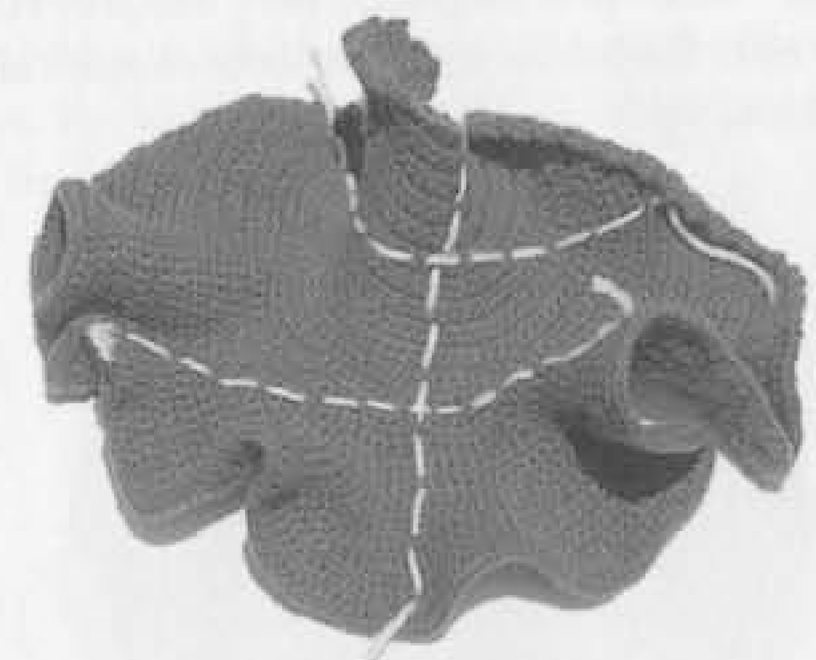
But Thurston's model is difficult to make, hard to handle, and inherently fragile.

Taimina intuited that the essence of this construction could be implemented with knitting or crochet by increasing the number of stitches in each row. As one increases, the surface naturally begins to ruffle and crenellate.

Having spent her childhood steeped in feminine handicrafts, Taimina first tried knitting. But the large number of stitches on the needles quickly became unmanageable and she soon realized that crochet offered a better approach.

The beauty of Taimina's method is that many of the intrinsic properties of hyperbolic space now become visible to the eye and can be directly experienced by playing with these models.

Geodesics, or straight lines, can be sewn onto the crochet texture for easy examination. Though the stitched lines in the model below appear curved, folding along them demonstrably produces a straight edge.

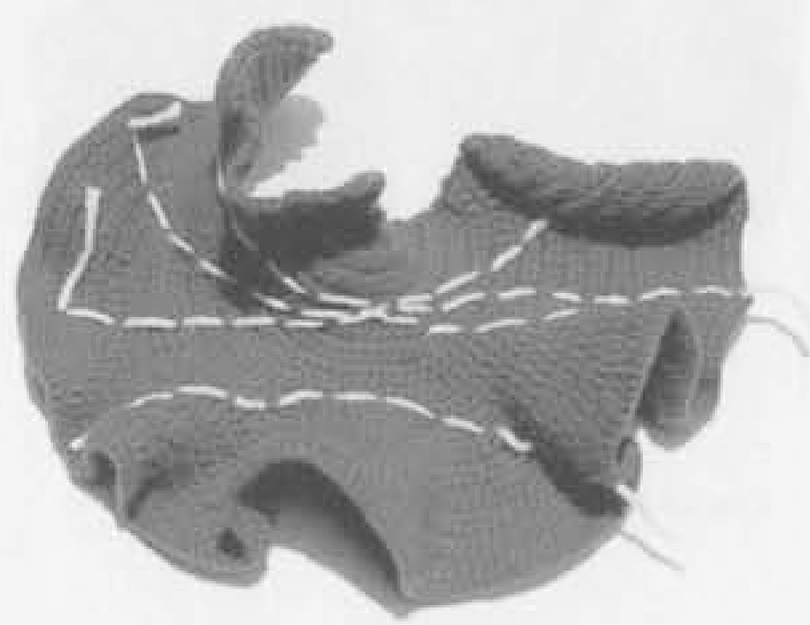


Crochet hyperbolic plane with geodesics stiched on.

Likewise, one can see immediately that the parallel postulate is violated.

In the model below there are three straight lines that pass through a point external to the bottom line. None of these upper lines ever intersect this original line. Handling this construction, you can physically fold along each line and verify materially the manifest untruth of Euclid's axiom.

Dr. Taimina's models are in demand from mathematics departments the world over and are featured in the Smithsonian's collection of American Mathematical Models. She and her husband, Dr. David Henderson, apply this woolly pedagogy in their courses at Cornell University and have promulgated her unique contribution to understanding non-Euclidean space in their classic college text book *Experiencing Geometry.*



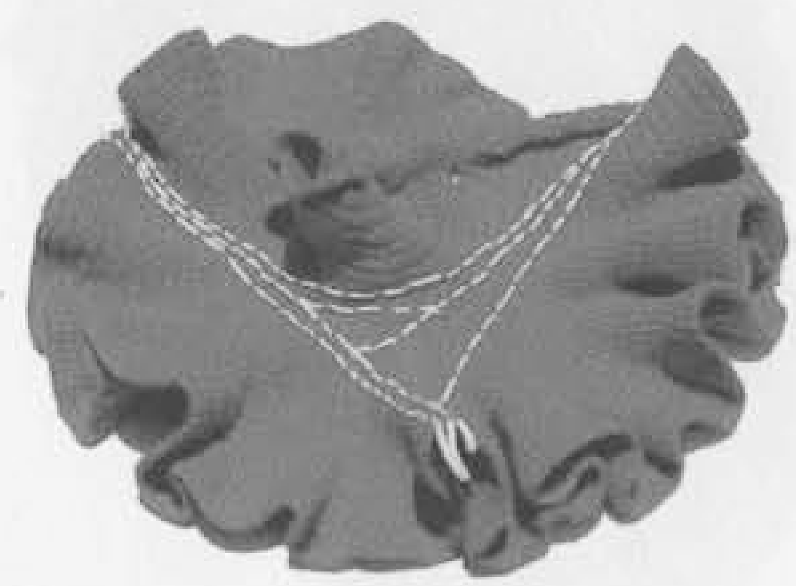
Crochet model demonstrating the falsity of Euclid's parallel postulate.

Further aspects of hyperbolic space that can be explored in a tactile way with Taimina's models are the properties of triangles.

In school we learn that the angles of a triangle always sum to 180°. While that is true on a Euclidean plane, it is not true on the surface of a sphere or on a hyperbolic plane.

On a sphere, the interior angles of a triangle sum to more than 180°—a fact you may verify for yourself by drawing on a balloon or a beach ball.

On a hyperbolic surface, triangular angles add to less than 180°. Moreover, the larger the triangle, the smaller the total angular sum. Until finally, when the triangle's points are infinitely far apart—making the largest possible three-pointed figure—the angles will sum to zero degrees. The angular oddity of this "ideal triangle" can be seen on the model below.



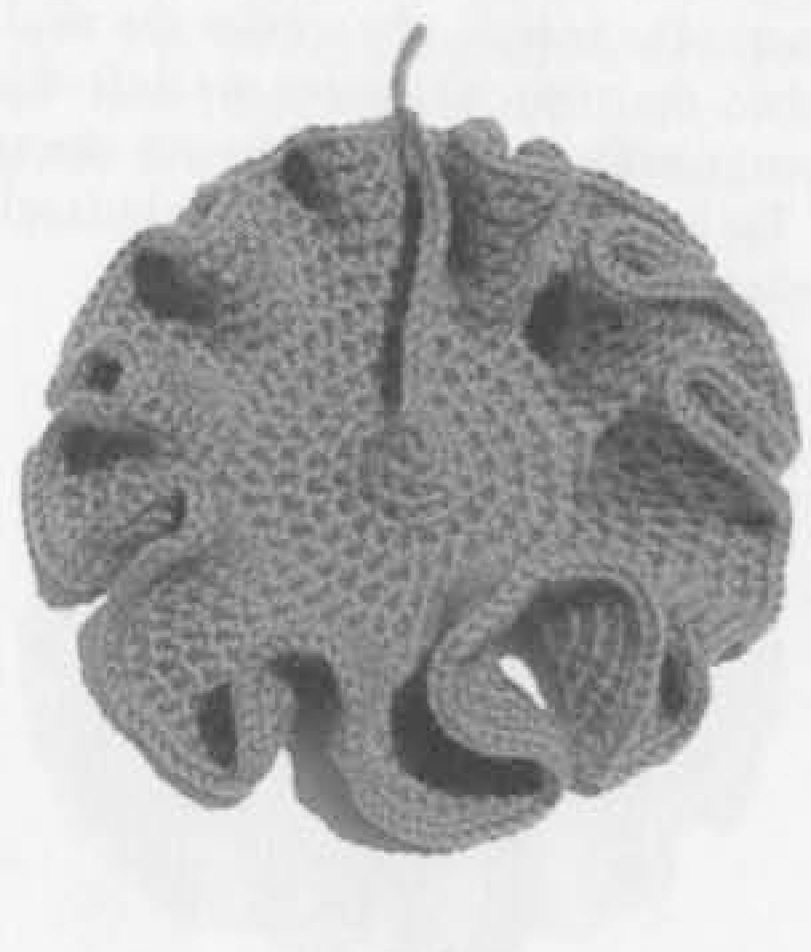
Crochet model showing an "ideal triangle"—whose angles sum to zero degrees.

Taimina's crochet models also enable us to make more exotic hyperbolic constructions.

Just as we can take a piece of paper and wrap it into a cone, so we can make a hyperbolic cone from a piece of hyperbolic paper.

The resulting form is a *pseudosphere.*

Again, the crenellated ruffles result from the process of continually increasing the number of stitches—this time we are crocheting in a spiral pattern. In the model below the rate of increase is one stitch in every three.



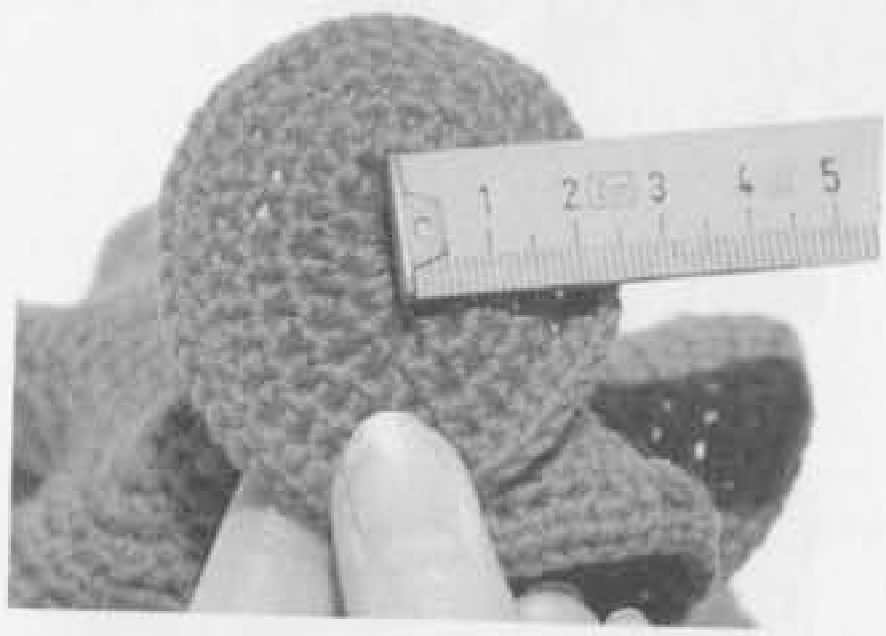
As the rate of increase of the stitches itself increases, the resulting construction becomes ever more crenellated. Whereas all spheres appear to have the same form, varying only in size, representations of hyperbolic surfaces may differ dramatically.

Here the rate of increase is one stitch in two.

Here, we are increasing in every stitch.

Mathematicians characterize such surfaces by the *radius* of the hyperbolic plane, the hyperbolic analog of a spherical radius. The flatter the surface, the greater its radius. A Euclidean plane may be understood as a hyperbolic plane of infinite radius.

The radius of any given hyperbolic plane is the radius of a circle within the plane that would sit flatly on a tabletop.



By using a very high rate of expansion—by increasing in every stitch, or even two or three times in every stitch—we can make a hyperbolic surface that starts from just a few stitches and expands out to an arbitrarily long perimeter.

Dr. Taimina's personal record is a plane that began with two dozen stitches and now has a perimeter of 369 inches. The width of this model is just 4 inches and the total weight of wool nearly a pound.

The mathematical discovery of hyperbolic space in the early nineteenth century raised the possibility that perhaps nature, also, had other options. What geometry was physical space?

For Isaac Newton and his contemporaries, the space of our universe was Euclidean—an endless and formless void. Most philosophers of the early modern period were so enamored of Euclidean form that Immanuel Kant argued the physical cosmos must *a priori* be this way.

But maybe space was some other, more interesting structure. Though Euclidean geometry may be easier for the human mind to grasp, that does not mean nature has chosen it for her ultimate temple. As Poincaré adroitly noted: "One geometry cannot be more true than another; it can only be more convenient."

In the late nineteenth century physicists tried to see if indeed they could detect a deviation from Euclidean norms by measuring the angles between stars. If cosmological space was not flat then the angles between three stars in a triangular configuration would not add up to 180°.

All measurements revealed the standard sum and for most of the past century the evidence has pointed to a Euclidean framework.

In the past two decades however, new discoveries have raised the possibility that our universe may be hyperbolic.

If that is so then it will not be the infinite void of classical physics, but a decidedly finite structure.

The idea of a finite hyperbolic space may sound improbable, for like the Euclidean plane the hyperbolic plane is endless. But just as we can construct finite Euclidean forms, so we can construct finite hyperbolic spaces.

Take a piece of paper and wrap it into a cylinder. Though the object you are holding is now curved, technically speaking it remains Euclidean because mathematically every tiny section is still formally flat.

Now imagine that you bend the cylinder itself, wrapping its ends around to connect with one another so the resulting form is a donut. If you actually try to do this, the paper will crease and the surface will buckle, but if we allow ourselves to make this move in a fourth dimension the resulting donut would have a perfectly flat surface.

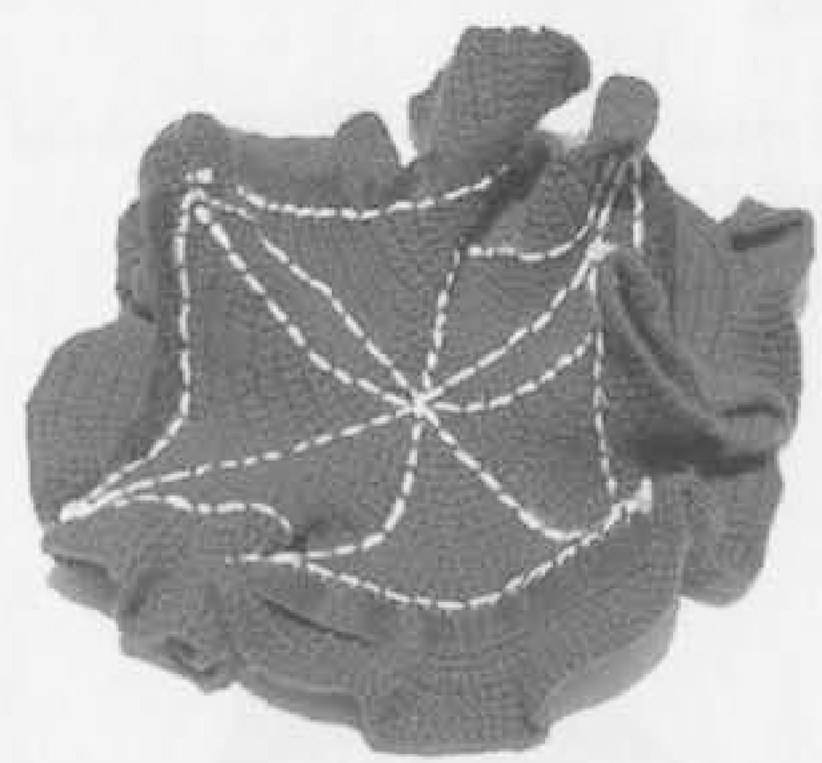
Such a torus is an example of a two dimensional space that is both finite and Euclidean.

Let us now consider a similar construction using hyperbolic paper?

Again, we can turn to Taimina's models for help in visualization.

We start with a piece of hyperbolic "paper"—cloth would perhaps be a better word—this time in the shape of an octagon.

Just as we wrapped our Euclidean paper into a cylinder by joining two opposite sides together, we do the same thing with our hyperbolic paper—only now we have *two* sets of opposite sides to connect.



The resulting form is a two-pronged cylinder, resembling a pair of trousers—the technical term is *hyperbolic pants.*

Just as our original cylinder was still Euclidean at every point, so we can prove that these pantaloons retain their hyperbolic geometry at each point.

Now we want to make the next move and wrap this construction into the hyperbolic equivalent of a torus. Again, if we try this in regular space the pant-legs would buckle and we'd lose the geometric smoothness. But as before, we can make the move by transitioning to a fourth dimension.

This time we'd create a double torus—or two-holed donut—that remains hyperbolic at every point.

If we now consider three-dimensional forms, rather than the two-dimensional surfaces we've been looking at so far, it turns out that the shape of our universe may be a kin to this hyperbolic structure.

Dr. Jeffrey Weeks, a maverick geometer and expert on hyperbolic space, has calculated that our universe may indeed have a finite geometry with a hyperbolic radius of 18 billion light years. In the next few years, the WMAP satellite currently taking pictures of the early universe may provide evidence one way or other, so that humanity may know at last the geometry of existence itself.