Statement of the Axiom of Choice:

Theorem 2.7 (b) If $\exists\, h: B \longrightarrow A \ni fh = I_B$, then f is onto. 满射

Note: The converse of (b), that $f: A \xrightarrow{\text{(onto)}} B$ implies $\exists\, h: B \longrightarrow A \ni fh = I_B$ can also be proved if the axiom of choice is assumed.

For each $b \in B$, let $A_b = \{x \in A \mid f(x) = b\}$. Then $A_b \neq \varnothing$ for all $b \in B$, since f is onto. Choose $a_b \in A_b$ and define $g(b) = a_b$ for each $b \in B$. Then $fg(b) = f(g(b)) = f(a_b) = b = I_B(b)$; hence, $fg = I_B$.

"Choose $a_b \in A_b$" 这句话要成立，就需要一个选择公理:

"选择"

(我的理解是对"随意选一个"有一种系统的说法)

即说明选择函数的存在性

Axiom ?

Axiom of Choice: Every family of non-empty has a choice function.

Given sets $\underset{\neq\varnothing}{S_\alpha}\ (\alpha \in \underset{\text{set}}{A})$, there exists $A \xrightarrow{f} \bigcup_{\alpha\in A} S_\alpha$ s.t. $f(\alpha) \in S_\alpha$ for all $\alpha \in A$.

extra：对于特定的集族 X，比如假设 X 是由自然数集合组成的集族。由于每个自然数的非空集合都有一个最小元，所以可以简单的挑选每个集合的最小元。

在能够指定一个明确选择方式的时候，选择公理是没有必要的。也就是说集族中存在具体规则，可以不应用选择公理。

什么叫做存在具体规则，也就是说如果能找到一个性质P，并且无穷多个集合的每个集合中有一个元素具有性质P，那就可以断言元素集的存在，这样就避开了无穷多次的具体选择。

extra：设有无穷多双鞋，想要建立一个集合，使它恰好含有每双鞋中的一只，这是不必进行无穷多次选择的。因为鞋分左右脚，只要取全部的左脚鞋就可以了。这个"左脚鞋"就是性质P。

但是如果是散乱的无穷多双袜子，上述方法就行不通了，袜子不分左右，找不到性质P把他们区分开。

By using the axiom of choice, we let θ be a choice function for the family of sets $\{A_b \mid b \in B\}$. We define $g(b) = \theta(A_b)$.

对这个公理的争议，因为它的使用所产生的集合似乎没有明确定义（即定义取决于选择）。然而，哥德尔的一个著名结果表明，自由使用选择公理不会导致矛盾，除非不使用选择公理就已经可以推导出矛盾。因此，如果我们假设除了选择公理之外我们的假设是一致的（即不会导致矛盾），那么就不可能从其他假设中反驳选择公理。1963 年，Paul Cohen 证明了选择公理也不能从其他假设中得到证明。

选择公理的第一个等价形式是良序定理.

我们对良序定理直观的表述为

Start by choosing any element $a \in X$, and let a be the smallest element of X. Next choose an element to be the smallest element of $X \setminus \{a\}$, and keep on going until the process cannot be continued. The process would never halt until X is exhausted; for if Y is the set of elements still unchosen after the process has stopped, one can resume the process by choosing a smallest element for the set Y. The order in which the elements of X were chosen would then be a well-ordering of X.

但这不是有效的证明.一方面,我们在做基于无限数量的选择,这种选择选择公理也不会给出我们这样的许可. 但在1904年, Zelmerlo 用选择公理给出了严格的证明.

两种能把对无限过程的模糊的想法转化为严格的证明的方法是一个非常重要的数学发展.

一种是基于良序原理的超限归纳法,我们将在第8章第5节讨论.

一种是Zorn 引理.

↓接下来证明选择公理的四种等价形式.

The definitions:

- Partial order
- Power set
- upper bound
- greatest element (in the chain)
- the least upper bound
- maximal element (in the set)
- linear
- well-ordered

Def. 4.1.

- Let X be a partially ordered set. A subset S is called a **chain** of X if the induced partial ordering on S is linear.

Def. 4.2.

A chain of a partially ordered set X is a **maximal chain** of X if it is not a proper subset of another chain of X.

Proof of Lemma 4.2:

We shall show that U has a fixed point under f, where a is given and U is defined above.

Let $\mathcal{A}$ be the collection of subsets S of U satisfying:

(1) $a \in S$

(2) $f(S) \subseteq S$

(3) If $S_0 \subseteq S$ and lub S_0 exists in X, then lub $S_0 \in S$.

It is easily proved that $U \in \mathcal{A}$. Next, we let $B = \cap \mathcal{A}$. It is easily proved that B is the smallest element of $\mathcal{A}$ (i.e. $B \in \mathcal{A}$, and if $S \in \mathcal{A}$, then $B \subseteq S$).

The aim will be to prove that B is a chain. For if this is done, then B would have a least upper bound, say, a_0. But then $a_0 \in B$ by condition (3) in the definition of $\mathcal{A}$. By condition (2), $f(a_0) \in B$; hence, $f(a_0) \leq a_0$ because a_0 is an upper bound for B. But it is given that $f(a_0) \geq a_0$; hence, $f(a_0) = a_0$.

$a < b < c$

$f(c) \leq b \leq f(b)$

$b \leq lub$

To prove that B is a chain, we consider the following sets:

(1) $a_0 \in C$, $f(C) \subseteq C$, $a_0 \geq f(c)$ $\therefore a_0 \in B(c)$

(2) $f(c) \leq b \leq f(b)$ $\therefore f(b) \in B(c)$

(3) $lub\, B(c) = a_0$, $a_0 \in B(c)$.

Let $C = \{c \in B \mid b < c \Rightarrow f(b) \leq c, \forall b \in B\}$

For each $c \in C$, let $B(c) = \{b \in B \mid b \leq c \text{ or } f(c) \leq b\}$. It is an exercise that $B(c) \in \mathcal{A}$; hence, since

B is the smallest element of $\mathcal{A}$

$B(c) \subseteq B$, we conclude that $B(c) = B$. Using this, it is an exercise to show that $C \in \mathcal{A}$, hence, that $C = B$.

$C \subseteq B$

(1) $a_0 \in C$, obviously.

(2) $b < c \leq f(c) \Rightarrow f(b) \leq c \leq f(c)$ $\therefore f(C) \subseteq C$.

(3) $f(a_0) = a_0 \in C$.

(2) $b < f(c)$

(3) $b < c < lub$

This shows that for b, c, either $b \leq c$ or $c \leq b$; hence, B is a chain.

Proof of Theorem 4.1:

The axiom of choice (1): Given sets $S_\alpha\,(\alpha\in A)$, there exists $A \xrightarrow{f} \bigcup_{\alpha\in A} S_\alpha$ s.t. $f(\alpha)\in S_\alpha$ for all $\alpha\in A$.

$S_\alpha \neq \emptyset$, A set

可推出↓

(1)' $\forall$ set X $\exists\, \underbrace{P_0(X)}_{=\{A \mid A\subseteq X,\, A\neq\phi\}} \xrightarrow{f} X$, $\forall S\in P_0(X)$, $f(S)\in S$.

(1)'⇒(4) Assume X has no maximal element.

若非空,则 a 为 maximal element

i.e. $\forall a\in X$, $F_a := \{x\in X \mid x>a\} \neq \phi$

由 Axiom of choice, $\exists\, \{F_a \mid a\in X\} \xrightarrow{\theta} X$ s.t. $\theta(F_a)\in F_a\subseteq X$.

Define a map $X \xrightarrow{f} X$, $f(a)=\theta(X_a)\in X_a$.

说明 $f(a)>a$.

则不会有 $f(a)=a$! 这与 Lemma 4.2 相矛盾.

也可以理解为
$$\begin{tikzcd}
F_a & \{F_a \mid a\in X\} \arrow[r, "\theta"] & X \\
a \arrow[u] & X \arrow[u] \arrow[ur, "g"'] &
\end{tikzcd}$$

(4)⇒(5) Let $\mathcal{B}$ be the collection of well-ordered subsets of X. Thus, $\mathcal{B}=\{(A,\leq) \mid A\subseteq X$, and $\leq$ well-orders $A\}$.

We partically order $\mathcal{B}$ by $\preccurlyeq$ where $(A_1,\leq_1)\preccurlyeq(A_2,\leq_2)$

(i) $A_1\subseteq A_2$

(2) The well-ordering $\leq_1$ coincides with the well-ordering on A_1 induced by $\leq_2$.

$\leq_2$ 作用在 A_1 上的结果与 $\leq_1$ 作用后的结果相同.

(iii) every member of A_2/A_1 is an upper bound of A_1.

Obviously, $\preccurlyeq$ is a partical ordering of $\mathcal{B}$.

<u>Claim</u>. Every chain of $(\mathcal{B},\preccurlyeq)$ has a lub.

Let $\mathcal{A}$ be a chain of $\mathcal{B}$, let $A_0=\cup\mathcal{A}$. A_0 is well-ordered by $\leq_0$ where for all $a,b\in A_0$ we set

不应该是∃

$a\leq_0 b$ iff $\exists (A_1,\leq_1)\in\mathcal{A}$, $a,b\in A_1$ and $a\leq_1 b$. 角标换成 i 更好? It now readily follows that

$(A_0, \leq_0)$ is a least upper bound for $\mathcal{A}$.

By (4), we find that $\mathcal{B}$ has a maximal element $(M, \leq)$.

<u>Claim.</u> $M = X$ and $\leq$ well orders X.

For if $M \neq X$, we can choose $x \in X \setminus M$ and extend $\leq$ to $M \cup \{x\}$ with $m \leq x$ for all $m \in M$. Then $(M \cup \{x\}, \leq) \in \mathcal{B}$ and $(M, \leq) \lneqq (M \cup \{x\}, \leq)$, contrary to the maximality of M.

甚至此处是$<$，因不等.

这一段详细地给"取它"一个描述

(可能? Remark 部分)

(5)⇒(1) Let $\mathcal{A}$ be a family of non-empty sets, and let $X = \cup \mathcal{A}$. From (5), we know that X is well-ordering. So for each $A \in \mathcal{A}$, define $\theta(A)$ as the least element of A. Then θ is a choice function of $\mathcal{A}$.

Because X is well-ordered and A is a subset of X, A is well-ordered as well.

(2)⇒(3) $\exists$ max. chain of M, m be the upper bound of M in X.

Then m is a maximal element in X; otherwise, $\exists x \in X$, $x > m$, then $M \cup \{x\}$ is a chain and $M \cup \{x\} \supsetneq M$

与 maximal chain 相反.

(4)⇒(2) Partically ordered set $\Rightarrow \mathcal{C} = \{C \mid C \text{ is a chain of } X \text{ w.r.t. } \leq\}$.

对此作 partically ordering: $\subseteq$

If $\mathcal{B}$ is a chain of $\mathcal{C}$ (chain's chain) $\longrightarrow \cup\mathcal{B}$ is the lub of $\mathcal{B}$. $\xrightarrow{(4)}$ we know that

① $\forall x \in \mathcal{B}$, $x \subseteq \cup\mathcal{B}$. $\therefore \cup\mathcal{B}$ is an upper bound.

比较 obvious ? $\cup\mathcal{B}$ is the least upper bound.

$\cup\mathcal{B}$ is the maximal element.

i.e. the maximal chain of X.

Exercises

Exercise 5: (5)⇒ $\forall$ surj. $X \xrightarrow{f} Y$ $\exists$ $Y \xrightarrow{g} X$ s.t. $f \circ g = id_Y$.

Choose a well-ordering $\leq$ on X. For $y \in Y$, define $g(y) :=$ the least element of $f^{-1}(y)$. $f \circ g(y) = y$